The Division Problem under Constraints*

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Abstract: The division problem under constraints consists of allocating a given amount of an homogeneous and perfectly divisible good among a subset of agents with single-peaked preferences on an exogenously given interval of feasible allotments. We characterize axiomatically the family of extended uniform rules proposed to solve the division problem under constraints. Rules in this family extend the uniform rule used to solve the classical division problem without constraints. We show that the family of all extended uniform rules coincides with the set of rules satisfying efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, consistency, and independence of irrelevant coalitions.

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1 Introduction

In the division problem an amount of a perfectly divisible good has to be allocated among a set of agents with single-peaked preferences on the set of all positive amounts of the good. An agent has a single-peaked preference if he considers that there is an amount of the good (the peak) strictly preferred to all other amounts and in both sides of the peak the preference is monotonic, decreasing at its right and increasing at its left. A profile is a vector of single-peaked preferences, one for each agent. It would then be desirable that the chosen vector of allotments of the good depended on the profile. But since preferences are idiosyncratic they have to be elicited by a rule selecting, for each profile of single-peaked preferences, a vector of allotments adding up to the total amount of the good. But in general, the sum of the peaks will be either larger or smaller than the total amount to be allocated. Then, a rule has to solve a positive or negative rationing problem, depending on whether the sum of the peaks exceeds or falls short the amount of the good. Rules differ from each other on how this rationing problem is resolved in terms of its induced properties like the strategic incentives faced by agents, efficiency, fairness, monotonicity, consistency, etc.

The literature on the division problem describes many examples of allocation problems that fit well with this general description. For instance, a group of agents participate in an activity that requires a fixed amount of labor (measured in units of time). Agents have a maximal number of units of time to contribute, and consider working as being undesirable. Suppose that labor is homogeneous and the wage is fixed. Then, strictly monotonic and quasi-concave preferences on the set of bundles of money and leisure generate single-peaked preferences on the set of potential allotments where the peak is the amount of working time associated to the optimal bundle. Similarly, a group of agents join a partnership to invest in a project (an indivisible bond with a face value, for example) that requires a fixed amount of money (neither more nor less). Their risk attitudes and wealth induce single-peaked preferences on the amount to be invested. Finally, a group of firms with different sizes have to jointly undertake a unique project of a fixed size. Since they may be involved in other projects their preferences are single-peaked on their respective allotments of the project. In all these cases, it is required that a rule solve the rationing problem arising from a vector of peaks that do not add up the needed amount. The uniform rule has emerged as a satisfactory way of solving the division problem. It tries to allocate the good as equally as possible keeping the bounds imposed by efficiency. Sprumont (1991) started a long list of axiomatic characterizations of the uniform rule by showing first that it is the unique efficient, strategy-proof and anonymous rule, and second that anonymity in this characterization can be replaced by envy-freeness. Ching (1994) strengthens the results of Sprumont (1991) by showing that the uniform rule is the only one satisfying efficiency, strategy-proofness and equal treatment of equals. Ching (1992), Dagan (1996), Schummer and Thomson (1997), Sönmez (1994), and Thomson (1994a, 1994b, 1995, and 1997) contain alternative characterizations of the uniform rule in the division problem. In the survey on strategy-proofness of Barberà (2010) the division problem and the uniform rule play a prominent role.

However, almost all the literature on the division problem has implicitly neglected the fact that in many applications (like those described above), agents' allotments may be constrained by objective and verifiable minimal and maximal capacities which impose lower and upper bounds on them. Those constraints may come from physical, legal or economic restrictions. Most often, real-life applications of the division problem have the feature that agents' allotments are constrained. For instance in problems where the good to be divided is time, like in the internal distribution of labor in a division of a firm, or in a bureau, or in a law firm, or like in the assignment of teaching duties among a given set of teachers of a particular subject in a school or university department. In all those cases, constraints due to physical or legal limitations (like labor contracts) impose unavoidable bounds to the agents' allotments. But constraints also show up in problems where agents have to contribute with money to finance a project of a fixed value, if they face budget constraints or if, due to implicit participation costs, their contributions have to be larger than a given amount (and hence, agents' allotments are bounded below as well). Big projects that can not be carried out by a single firm may be split among a set of firms which are not able to undertake alone the project precisely due to their capacity constraints and, in addition, each firm participation (in order to be valuable to itself) may require receiving an allotment of the project above a given amount. Therefore, in all these cases the division problem is restricted further by feasibility constraints that are described by a family of closed intervals of non-negative feasible allotments, one for each agent. It is then natural to assume that each agent has a closed interval of feasible allotments and his idiosyncratic preferences are single-peaked on this interval. Moreover, we will be interested in situations where agents' participation is voluntary; namely, each agent has to consider all his strictly positive feasible allotments as being strictly preferred to receive zero (the allotment associated to the prospect of non-participating in the division problem). What is specific to our paper is that we assume that each agent's allotment has to either belong to a given feasible interval of allotments or else be equal to zero. Hence, a division problem under constraints is composed by the set of agents, the amount of the good to be allocated among them, the vectors of lower and upper bounds of their feasible intervals, and their single-peaked preferences on their respective feasible intervals. We want to emphasize that our model contains all particular instances of division problems where agents' allotments are *only* constrained by maximal capacity restrictions because the lower bounds may be equal to zero, as in some of the real-life applications that we have just described above.

Given a division problem under constraints, it may be the case that there does not exist a vector of feasible allotments adding up to the total amount to be allocated. Hence a rule has two components. First, the choice of an admissible and non-empty subset of agents among whom it is possible to allocate the amount of the good keeping their feasibility constrains; if there is no such subset, then the rule has to choose the zero allotment for all agents. Second, and given this chosen admissible non-empty subset of agents (called participants), the rule has to assign to each participant a feasible allotment in such a way that their sum adds up to the total amount to be allocated.

Our contribution in this paper is to define extensions of the uniform rule to this class of division problems under constraints and to provide an axiomatic characterization of them by using two classes of desirable properties. The first class is related to the behavior of the rule at a given division problem under constraints. First, *efficiency.* A rule is efficient if it always selects Pareto optimal allocations. Second, *equal treatment of equals.* A rule satisfies equal treatment of equals if identical participants receive the same allotment. The second class is related to the restrictions that the properties impose on a rule when comparing its proposal at different division problems under constraints. First, *strategy-proofness.* A rule is strategy-proof if no agent can profitably alter the rule's choice by misrepresenting his preferences. Second, *bound monotonicity.* Assume that the upper bound of an agent decreases. Two situations are possible. Either the allotment of this agent in the initial problem is not larger than the new upper bound or it is strictly larger than the new upper bound. In the first situation bound monotonicity says that both problems must have the same allotment. In the second situation, bound monotonicity says that the agent must receive his new upper bound whereas the rest of the agents can not receive smaller allotments. Symmetric arguments can be applied when the lower bound of an agent increases. Third, *consistency.* Assume that after applying the rule to a given problem a subset of agents leave with their assigned allotments. Consider the new problem with the remaining set of agents and the total amount of the good minus the sum of the allotments received by the agents that already left. The rule is consistent if the allotments it proposes to the remaining agents in the reduced problem coincides with their allotments in the original problem. Fourth, *independence of irrelevant coalitions.* Assume the set of admissible coalitions in one problem is contained in the set of admissible coalitions in another problem and the coalition chosen by the rule in the larger problem is admissible for the smaller one, then this property says that the rule has to select the same coalition of participants in the two problems.

The main findings of the paper appear in two theorems. In Theorem 1 we show that in the subclass of division problems under constraints with the property that the full set of agents is admissible, the *feasible uniform rule* is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity. This result is an extension of the characterization of Ching (1994) for the uniform rule in the classical division problem. The feasible uniform rule on this subclass of division problems under constraints tries to allocate the good among all agents in the most egalitarian way respecting not only the bounds imposed by efficiency, but also those imposed by the feasibility constraints. An *extended uniform rule* on the class of all division problems under constraints selects first, using a monotonic and responsive order on the family of all non-empty and finite subsets of agents, an admissible coalition of participants (if any, otherwise it chooses the zero allotment for all agents) and then it applies the feasible uniform rule to the reduced division problem under constraints obtained by restricting the original problem to this admissible subset of participants. We show in Theorem 2 that the class of all extended uniform rules (each one associated to a monotonic and responsive order on the non-empty and finite subsets of agents) coincides with the set of rules satisfying efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, consistency and independence of irrelevant coalitions.

Several papers are closely related to the present one. First, Bergantiños, Massó and Neme (2012a) studies the division problem with maximal capacity constraints under the assumption that the sum of all upper bounds is larger than the total amount of the good that has to be distributed. Second, Kibris (2003) studies the division problem with maximal capacity constraints assuming free-disposability of the good. Then a rule assigns to each division problem with maximal capacity constraints a vector of allotments satisfying the constraints and adding up *less* or equal than the total amount. Kibris (2003) characterizes an extension of the uniform rule to his setting with free-disposability. Third, Bergantiños, Massó and Neme (2012b) considers the division problem when agents' participation is voluntary. Each agent has an idiosyncratic interval of acceptable allotments (which, in contrast with our setting here, is private information) where his preferences are single-peaked. Then a rule proposes to each agent either to not participate at all or an acceptable allotment. Bergantiños, Massó and Neme (2012b) shows that strategy-proofness is too demanding in this setting. Then, they study a subclass of efficient and consistent rules and characterize extensions of the uniform rule that deal explicitly with agents' voluntary participation. Fourth, Kim, Bergantiños and Chun (2012) characterize two families of rules, related with the rules studied in Bergantiños, Massó and Neme (2012b) and this paper, using the separability principle and other properties. Fifth, Manjunath (2012) proposes a division problem where each agent's preferences are characterized by a top and a *minimum* allotment in such a way that the agent is indifferent between any two quantities that are either below the minimum acceptable allotment or above the top allotment. Manjunath (2012) first shows that, under different fairness properties, strategy-proofness and efficiency are incompatible and second, he characterizes axiomatically different rules that solve the rationing problem in his setting. Finally, the division problem with maximal capacity constraints is also considered by Moulin (1999).[1] He characterizes the class of all fixed path mechanisms as the set of rules satisfying efficiency, strategy-proofness, consistency and resource monotonicity. Ehlers (2002a) presents a shorter proof of the main result in Moulin (1999) and Ehlers (2002b) extends it by showing that, for problems with strictly more than two agents, the class of all fixed path mechanisms coincides with the set of rules satisfying weak one-sided resource monotonicity, strategy-proofness and consistency.

The paper is organized as follows. In Section 2 we describe the model. In Section 3 we define several desirable properties that a rule may satisfy. In Section 4 we define the feasible uniform rule (for the subclass of division problems under constraints where the grand coalition is admissible) and the extended uniform rule induced by a monotonic and responsive order on the family of all finite and non-empty subsets of agents and state their axiomatic characterizations. Section 5 contains some final remarks stating other desirable properties that all extended uniform rules also satisfy. The proofs are in Section 6.

2 Preliminaries

Let $t > 0$ be an amount of an homogeneous and perfectly *divisible good*. A finite set of *agents* is considering the possibility of dividing t among a subset of them, to be determined according to their preferences. We will consider situations where the amount of the good t and the finite set of agents may vary. Let $\mathbb{N}$ be the set of positive integers and let $\mathcal{N}$ be the family of all non-empty and finite subsets of $\mathbb{N}$. The set of agents is then $N \in \mathcal{N}$ with cardinality n. In contrast with Sprumont (1991), we consider decision problems where the amount of the good received by each agent $i \in N$ is constrained either to belong to a given closed interval $[l_i, u_i] \subseteq [0, +\infty)$, determined by lower and upper exogenous constraints (l_i and u_i, respectively), or to be equal to zero. That is, an agent is either excluded from the

[1] In Moulin (1999) the maximal capacity constraints are justified on the basis of technical simplicity in oder to define the priority rationing methods by an ordinary path and to define the duality operator that cuts the main proof in half.

division (and receives zero) or else his allotment has to be feasible. We are interested in settings where the participation of the agents in the division problem is voluntary in the sense that all strictly positive feasible allotments are strictly better than receiving zero. Thus, agent i's preferences $\succeq_i$ are defined on the set $\{0\} \cup [l_i, u_i]$, with $0 \leq l_i \leq u_i \leq +\infty$ and $l_i < +\infty$. The set $[l_i, u_i]$ is agent i's *interval of feasible allotments*. We assume that $\succeq_i$ is a complete, reflexive, and transitive binary relation on $\{0\} \cup [l_i, u_i]$. Given $\succeq_i$, let $\succ_i$ be the antisymmetric binary relation induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \{0\} \cup [l_i, u_i]$, $x_i \succ_i y_i$ if and only if $y_i \succeq x_i$ does not hold) and let $\sim_i$ be the indifference relation induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \{0\} \cup [l_i, u_i]$, $x_i \sim_i y_i$ if and only if $x_i \succeq_i y_i$ and $y_i \succeq_i x_i$). We will also assume that $\succeq_i$ is single-peaked on $[l_i, u_i]$ and we will denote by $p_i \in [l_i, u_i]$ agent i's *peak*. Formally, agent i's preferences $\succeq_i$ is a complete preorder on the set $\{0\} \cup [l_i, u_i]$ that satisfies the following additional properties:

(P.1) there exists $p_i \in [l_i, u_i]$ such that $p_i \succ_i x_i$ for all $x_i \in [l_i, u_i] \backslash \{p_i\}$;

(P.2) $x_i \succ_i y_i$ for any pair of allotments $x_i, y_i \in [l_i, u_i]$ such that either $y_i < x_i \leq p_i$ or $p_i \leq x_i < y_i$; and

(P.3) $x_i \succ_i 0$ for all $x_i \in [l_i, u_i] \backslash \{0\}$.

Observe that agent i's preferences are defined on $\{0\} \cup [l_i, u_i]$ and are independent of t. Moreover, we are admitting the possibilities that $l_i = 0$ and $l_i = p_i = u_i$. Conditions (P.1) and (P.2) are the standard single-peaked restrictions on $[l_i, u_i]$ while condition (P.3) conveys the minimal voluntary participation requirement that all strictly positive allotments in the feasible interval are strictly preferred to the zero allotment. A preference $\succeq_i$ of agent i is (partly) characterized by the triple (l_i, p_i, u_i). There are many preferences of agent i with the same (l_i, p_i, u_i); however, they differ only on how two allotments on different sides of p_i are ordered while all of them coincide on the ordering on the allotments on each of the sides of p_i. This multiplicity will often be irrelevant. We will assume throughout the paper that for any agent i, the bounds l_i and u_i are fixed and exogenously given while the preference $\succeq_i$ over the interval $[l_i, u_i]$ is idiosyncratic and has to be elicited through a direct revelation mechanism. As we have already discussed in the Introduction, we are interested in division problems where allotments may be restricted by objective feasibility or capacity constraints while every preference $\succeq_i$ satisfying (P.1), (P.2), and (P.3) is a legitimate one for agent i.[2]

[2]See Bergantiños, Massó, and Neme (2012b) for an analysis of efficient and consistent rules in the division problem when the interval $[l_i, u_i]$ is the set of idiosincratic acceptable allotments for agent i and participation is voluntary.

Let $N \in \mathcal{N}$ be a set of agents. A *profile* $\succeq_N = (\succeq_i)_{i \in N}$ is an $n-$tuple of preferences satisfying properties (P.1), (P.2) and (P.3) above. Given a profile $\succeq_N$ and agent i's preferences $\succeq_i'$ we denote by $(\succeq_i', \succeq_{N \setminus \{i\}})$ the profile where $\succeq_i$ has been replaced by $\succeq_i'$ and all other agents have the same preferences. When no confusion arises we denote the profile $\succeq_N$ by $\succeq$.

A division problem under constraints (a *problem* for short) is a $5-$tuple $P = (N, t, l, u, \succeq)$ where $N \in \mathcal{N}$ is the finite set of agents, t is the amount of the good to be divided, $l = (l_i)_{i \in N}$ is the vector of lower constraints, $u = (u_i)_{i \in N}$ is the vector of upper constraints, and $\succeq$ is a profile. Although the vector of lower and upper constraints are part of the definition of the profile $\succeq$, for convenience we explicitly include them in the description of a problem. Let $\mathcal{P}$ be the set of all problems.

Given a problem $P = (N, t, l, u, \succeq)$ we denote by $P \backslash l_i'$ the problem obtained from P by replacing l_i by l_i' and such that the preferences of agent i on $[\max\{l_i, l_i'\}, u_i]$ coincide in both problems. Similar notation is used for $P \backslash u_i'$, $P \backslash l'$, $P \backslash \succeq_i$ and so on. Besides given $P \in \mathcal{P}$ and $S \subseteq N$, we denote by P_S the problem P when considering only agents in S; namely, $P_S = (S, t, (l_i)_{i \in S}, (u_i)_{i \in S}, (\succeq_i)_{i \in S})$.

A problem where all agents have single-peaked preferences on $[0, +\infty)$ is known as the *division problem*; *i.e.*, for all $i \in N$, $l_i = 0$, $u_i = +\infty$, and (P.1) and (P.2) hold.

The set of *feasible allocations* of problem P is

$$FA(P) = \left\{ (x_i)_{i \in N} \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \in \{0, t\} \text{ and, for each } i \in N, \ x_i \in \{0\} \cup [l_i, u_i] \right\}.$$

This set is never empty since the allocation $(0, ..., 0) \in \mathbb{R}_+^N$ is always feasible. Besides, there are problems for which $(0, ..., 0)$ is the unique feasible allocation.

A coalition $S \subseteq N$ is *admissible* (at problem P) if either S is empty or it is feasible to divide t among all agents in S; namely, coalition $S \neq \emptyset$ is *admissible* if there exists $x = (x_i)_{i \in S} \in \mathbb{R}_+^S$ such that $\sum_{i \in S} x_i = t$ and $l_i \leq x_i \leq u_i$ for all $i \in S$. Hence, $S \neq \emptyset$ is admissible if and only if $\sum_{i \in S} l_i \leq t \leq \sum_{i \in S} u_i$. We denote by $A(P)$ the set of all admissible coalitions at problem P. The set $A(P)$ is non-empty because it always contains the empty coalition.

A *rule* f assigns to each problem $P \in \mathcal{P}$ a feasible allocation; that is, $f(P) \in FA(P)$ for all $P \in \mathcal{P}$. Hence, a rule f can be seen as a systematic way of assigning to each problem $P \in \mathcal{P}$ two different but related aspects of the solution of the problem.

First, an admissible coalition $c^f(P) \in A(P)$ where

$$c^f(P) = \{i \in N \mid f_i(P) \in [l_i, u_i]\}.$$

We refer to the agents in $c^f(P)$ as *participants*. Often, and when no confusion arises because the problem P will be obvious from the context we write c^f instead of $c^f(P)$. Obviously, if $i \notin c^f(P)$, then $f_i(P) = 0$. Besides, if $l_i = 0$, then $i \in c^f(P)$.

Second, how the amount t is divided among the participants; *i.e.*, if $c^f(P) \neq \emptyset$ then,

$$\sum_{i \in c^f(P)} f_i(P) = t.$$

We will see later that to identify rules satisfying appealing properties we may have some freedom when choosing one among all admissible coalitions while the properties will determine a unique way of dividing the amount of the good among the participants.

3 Properties of Rules

In this section we define several properties that a rule may satisfy. The first four are basic and standard properties already used in many axiomatic analysis of the division problem. The last two are bound monotonicity, which restricts how the rule should change when the upper or lower bound of an agent changes, and independence of irrelevant coalitions, which restricts how the participants should be chosen.

A rule is efficient if it always selects a Pareto optimal allocation.

Efficiency (*ef*) For each $P \in \mathcal{P}$ there is no $(y_i)_{i \in N} \in FA(P)$ with the property that $y_i \succeq_i f_i(P)$ for all $i \in N$ and $y_j \succ_j f_j(P)$ for some $j \in N$.

Rules require each agent to report a single-peaked preference on $\{0\} \cup [l_i, u_i]$. A rule is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully; namely, truth-telling is a weakly dominant strategy in the direct revelation game induced by the rule.

Strategy-proofness (*sp*) For each $P \in \mathcal{P}$, $i \in N$, and $\succeq'_i$ on $\{0\} \cup [l_i, u_i]$,

$$f_i(P) \succeq_i f_i(P \backslash \succeq'_i).$$

Given a problem P we say that agent $i \in N$ *manipulates* f *at profile* $\succeq$ *via* $\succeq'_i$ if $f_i(P \backslash \succeq'_i) \succ_i f_i(P)$.

A rule satisfies strong equal treatment of equals if identical agents receive the same allotment.

Strong equal treatment of equals (*sete*) For every $P \in \mathcal{P}$ such that there are $i, j \in N$, $i \neq j$, and $\succeq_i = \succeq_j$ then, $f_i(P) = f_j(P)$.

Strong equal treatment of agents is incompatible with efficiency. To see that, consider any problem P where $N = \{1, 2, 3\}$, $t = 10$, $(l_i, p_i, u_i) = (4, 5, 10)$ for $i = 1, 2, 3$, and $\succeq_1 = \succeq_2 = \succeq_3$. Since the allotment $(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}) \notin FA(P)$ any f satisfying strong equal treatment of equals has the property that $c^f = \emptyset$ and $f_i(P) = 0$ for all $i = 1, 2, 3$. However, $(0, 5, 5)$ Pareto dominates $(0, 0, 0)$. Thus efficiency and strong equal treatment of equals are incompatible. For this reason, we restrict our attention to the weaker notion of the property requiring that only equal *participants* must be treated equally. The example above suggests that a rule satisfying equal treatment of equal (participants) will have to use some criteria to select among the three allotments $(0, 5, 5)$, $(5, 0, 5)$, and $(5, 5, 0)$ (and corresponding set of participants); but we will deal with that later.

A rule satisfies equal treatment of equals if identical participants receive the same allotment.

EQUAL TREATMENT OF EQUALS (*ete*) For every $P \in \mathcal{P}$ such that there are $i, j \in N$, $i \neq j$, $\succeq_i = \succeq_j$, and $i, j \in c^f(P)$ then, $f_i(P) = f_j(P)$.

We note that (*sete*) and (*ete*) coincide with the standard property of equal treatment of equals when they are applied to classical division problems.

A rule is consistent if the following requirement holds. Apply the rule to a given problem and assume that a subset of agents leave with their corresponding allotments. Consider the new problem formed by the set of agents that remain with the same preferences that they had in the original problem and the total amount of the good minus the sum of the allotments received by the subset of agents that already left. Then, the rule does not require to reallocate the allotments of the remaining agents.

CONSISTENCY (*cons*) For each problem $P \in \mathcal{P}$, each non-empty subset of agents $S \subseteq N$, and each $i \in S$,

$$f_i(P) = f_i\left(S, t - \sum_{j \in c^f(P) \setminus S} f_j(P), (l_i)_{i \in S}, (u_i)_{i \in S}, (\succeq_i)_{i \in S}\right).$$

We now introduce the property of bound monotonicity, which imposes restrictions on how the rule changes when the upper or lower bounds of the interval of feasible allotments of one agent changes. Take a problem P where the upper bound of agent k decreases to $u'_k < u_k$ without changing his preferences (*i.e.*, $\succeq'_k$ coincides with $\succeq_k$ on $[l_k, u'_k]$). A natural notion of bound monotonicity says the following. First, assume that $f_k(P) \leq u'_k$; then, $f(P)$ is also feasible in $P \backslash u'_k$. Bound monotonicity says that f selects the same allocation in both problems (*i.e.*, $f(P \backslash u'_k) = f(P)$). Second, assume that $u'_k < f_k(P)$; then, $f(P)$ is not feasible in $P\backslash u'_k$. If we can divide t in $P\backslash u'_k$ among the same set of agents as in P (*i.e.*, $c^f(P) \in A(P\backslash u'_k)$) then, bound monotonicity says that agent k receives his new upper bound ($f_k(P\backslash u'_k) = u'_k$) and the rest of agents receive an allotment that is not smaller than the one they received in P (*i.e.*, $f_i(P\backslash u'_k) \geq f_i(P)$ for all $i \in N\backslash\{k\}$). If u'_k is so small that we can not divide t in $P\backslash u'_k$ among the same set of agents as in P (*i.e.*, $c^f(P) \notin A(P\backslash u'_k)$) then, bound monotonicity says nothing. We apply the same ideas to the lower bound.

We now define the property of bound monotonicity formally.

Bound monotonicity (*bm*)

(bm.1) Let $P, P\backslash u'_k \in \mathcal{P}$ be such that $u'_k < u_k$, and $c^f(P) \in A(P\backslash u'_k)$. Then, $c^f(P\backslash u'_k) = c^f(P)$ and

$$f_i(P\backslash u'_k) \geq \min\{f_i(P), u'_i\} \text{ for each } i \in N, \tag{1}$$

where $u'_i = u_i$ for all $i \in N\backslash\{k\}$.

(bm.2) Let $P, P\backslash l'_k \in \mathcal{P}$ be such that $l_k < l'_k$, and $c^f(P) \in A(P\backslash l'_k)$. Then, $c^f(P\backslash l'_k) = c^f(P)$ and

$$f_i(P\backslash l'_k) \leq \max\{f_i(P), l'_i\} \text{ for each } i \in N, \tag{2}$$

where $l'_i = l_i$ for all $i \in N\backslash\{k\}$.

The property of bound monotonicity can be seen also as a weak property of solidarity. Thomson (1994b) says: "A condition that is natural however is that agents all lose together or all gain together when the amount to divide increases, in fact when it increases *or* decreases. The general requirement that all agents be affected in the same direction "as their environment changes" is the essence of solidarity." We can apply this solidarity principle when the environment changes because the upper bound of some agent changes (the case of a change in the lower bound is analogous). Take a problem P where the upper bound of agent k decreases to $u'_k < u_k$ without changing his preferences. First, assume that $f_k(P) \leq u'_k$ then, $f(P)$ is also feasible in $P\backslash u'_k$. Then, we select the same allocation in both problems (in this case we do the same as with (bm)). Second, assume that $u'_k < f_k(P)$ then, $f(P)$ is not feasible in $P\backslash u'_k$. If $c^f(P) \in A(P\backslash u'_k)$ then, agent k receives his new upper bound ($f_k(P\backslash u'_k) = u'_k$) and the rest of agents either all are better off or all are worse off. Namely, either

$$f_i(P\backslash u'_k) \succeq_i f_i(P) \text{ for each } i \in N\backslash\{k\} \text{ or}$$

$$f_i(P) \succeq_i f_i(P\backslash u'_k) \text{ for each } i \in N\backslash\{k\}.$$

Obviously, bound monotonicity does not imply solidarity and solidarity does not imply bound monotonicity. Nevertheless if a rule satisfy (ef), then solidarity implies (bm) but the other implication does not hold.[3] Thus, we can see (bm) as a weaker version of solidarity.

A rule satisfies independence of irrelevant coalitions if the following requirement holds. Consider two problems where the set of admissible coalitions of the first one is contained in the set of admissible coalitions of the second one. Assume that the coalition chosen by the rule in the second problem is admissible for the first one. Then, the rule chooses the same coalition of participants in the two problems. This property is inspired in the well-known principle of independence of irrelevant alternatives. Nash (1950) defined it, in bargaining problems, as follows. Suppose that the set of admissible outcomes of the bargaining problem S' is a subset of the set of admissible outcomes of the bargaining problem S. Besides, the solution of S belongs to S'. Then, the solution of S' must be the solution of S. Notice that we are just applying the same principle to the function c^f.

Independence of irrelevant coalitions (iic) For any two problems $P,\ P' \in \mathcal{P}$ such that $c^f(P) \in A(P') \subseteq A(P)$ then,

$$c^f(P') = c^f(P).$$

4 The Uniform Principle: Two Characterizations

In this section we present the two main results of the paper.

The uniform rule U on problems without constraints (see Sprumont (1991)) tries to allocate the good as equally as possible, keeping the efficiency bounds binding (all agents have to be rationed in the same direction). The feasible uniform rule, on the subclass of division problems under constraints with the property that the set of all agents is an admissible coalition, does the same than U but it takes also into account the feasibility constraints. We show in Theorem 1 that the feasible uniform rule is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity on this subclass of problems.

Let $\mathcal{P}^*$ be the set of division problems under constraints with the property that the set of all agents is an admissible coalition; namely,

$$\mathcal{P}^* = \left\{ P \in \mathcal{P} \mid \sum_{i \in N} l_i \leq t \leq \sum_{i \in N} u_i \right\}.$$

[3] We omit the non trivial proof of this statement.

Feasible uniform rule The *feasible uniform rule* U on $\mathcal{P}^*$ is defined as follows. For each $P \in \mathcal{P}^*$ and $i \in N$,

$$U_i(P) = \begin{cases} \min\{p_i, \max\{l_i, \alpha\}\} & \text{if } \sum\limits_{j \in N} p_j \geq t \\ \max\{p_i, \min\{u_i, \alpha\}\} & \text{if } \sum\limits_{j \in N} p_j < t, \end{cases}$$

where α is the unique number satisfying $\sum_{j \in N} U_j(P) = t$.

Remark 1 Consider the problem $P = (N, t, l, u, \succeq) \in \mathcal{P}^*$ and a division problem without constraints $(N, t, \succeq')$ (*i.e.*, $l_i' = 0$ and $u_i' = +\infty$ for all $i \in N$) such that for each $i \in N$, $\succeq_i'$ coincides with $\succeq_i$ on $[l_i, u_i]$ and $U(N, t, \succeq') \in FA(P)$. Then, $U(N, t, \succeq') = U(P)$. Thus, the feasible uniform rule U can be considered as an extension of the uniform rule from division problems without constraints to $\mathcal{P}^*$. Observe that the extension of the uniform rule to problems with voluntary participation presented in Bergantiños, Massó, and Neme (2012b) does not have this property. Let us clarify this with an example. Suppose that $N = \{1, 2\}$, $t = 10$, $l = (1, 3)$, $u = (8, 8)$ and $p = (6, 6)$. Thus, $U(P) = (5, 5)$ whereas the rule in Bergantiños, Massó, and Neme (2012b) chooses $(4, 6)$; namely, it increases uniformly the allotments starting from l.

Theorem 1 in Ching (1994) provides a characterization of the uniform rule in the classical division problem with (*ef*), (*ete*), and (*sp*). In Theorem 1 below we prove that if we add (*bm*) we have a characterization of the feasible uniform rule in $\mathcal{P}^*$. Thus our result can be seen as an extension of Ching's result.

Theorem 1 *The feasible uniform rule U on $\mathcal{P}^*$ is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity. Besides, the four properties are independent.*

Proof See Subsection 6.1.

Remark 2 Unfortunately, U does not satisfy solidarity on $\mathcal{P}^*$. Consider a problem $P \in \mathcal{P}^*$ where $N = \{1, 2, 3\}$, $t = 15$, $l = (0, 0, 0)$, $u = (10, 10, 10)$, $p = (6, 6, 6)$, $5 \succ_2 7$ and $7 \succ_3 5$. Thus, $U(P) = (5, 5, 5)$. Let $u_1' = 1$. Then $p_1' = 1$ and hence $p_1' + p_2 + p_3 = 13$. Now $U(P \backslash u_1') = (1, 7, 7)$, which means that U does not satisfy solidarity.

Therefore, it is not possible to use solidarity instead of (*bm*) in our results. Assume that a rule f satisfies solidarity, (*ef*), (*ete*), and (*sp*). Thus, f also satisfies (*bm*), (*ef*), (*ete*), and (*sp*). By Theorem 1, $f = U$, which is a contradiction because U does not satisfy solidarity. Hence, the properties of (*ef*), (*sp*), (*ete*), and solidarity are incompatible in $\mathcal{P}^*$. This fact is not surprising because solidarity is incompatible with some properties in the classical division problem, see for instance Thomson (1994b). The example suggests that in our model the incompatibility comes mainly from (*sp*). As Lemma 1.2 will establish, (*ef*) and (*sp*) imply own peak monotonicity while solidarity requires the use of the whole preferences.

We now consider the general case. We first extend the feasible uniform rule to $\mathcal{P}$. Let P be a problem in $\mathcal{P}$. An extended uniform rule selects at P the feasible set of participants by maximizing a given order ρ (a complete, antisymmetric and transitive binary relation) on $\mathcal{N}$, restricted to the family of admissible coalitions $A(P) \subseteq \mathcal{N}$, and then it applies the feasible uniform rule to this selected set of participants to choose their allotments.

Extended Uniform Rule Let ρ be an order on $\mathcal{N}$. The *extended uniform rule on* $\mathcal{P}$ *induced by the order* ρ *on* $\mathcal{N}$, denoted by U^{ρ}, is defined as follows. For each $P \in \mathcal{P}$ and $i \in N$,

$$U_i^{\rho}(P) = \begin{cases} U_i(P_{c^{U^{\rho}}(P)}) & \text{if } i \in c^{U^{\rho}}(P) \\ 0 & \text{if } i \notin c^{U^{\rho}}(P), \end{cases}$$

where $c^{U^{\rho}}(P) \in A(P)$ and $c^{U^{\rho}}(P)\, \rho S$ for all $S \in A(P) \backslash c^{U^{\rho}}(P)$.

Obviously, the family of extended uniform rules on $\mathcal{P}$ is large. We are interested in the subfamily of rules that satisfy efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, consistency and independence of irrelevant coalitions. To identify it we restrict the order ρ on $\mathcal{N}$ to satisfy the properties of monotonicity and responsiveness.

Definition 1 We say that an order ρ on $\mathcal{N}$ is

(i) *monotonic* if for all $S \in \mathcal{N}$ and $i \notin S$, $(S \cup \{i\})\, \rho S$; and

(ii) *responsive* if for all $S, T \in \mathcal{N}$ and $i \notin S \cup T$, $S \rho T$ implies $(S \cup \{i\})\, \rho\, (T \cup \{i\})$.

If ρ is monotonic, then $c^{U^{\rho}}$ is maximal. Namely, if $c^{U^{\rho}}(P) \subsetneq S$, then S is not admissible.

Theorem 2 below characterizes the set of extended uniform rules that choose the admissible coalition according to a monotonic and responsive order ρ on $\mathcal{N}$. The way in which we obtain ρ is similar to the one used in Bergantiños, Massó, and Neme (2012b).

Theorem 2 *Let f be a rule on $\mathcal{P}$. Then, f satisfies efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, consistency, and independence of irrelevant coalitions if and only if $f = U^{\rho}$ for some monotonic and responsive order ρ on $\mathcal{N}$. Besides, the six properties are independent.*

Proof See Subsection 6.2.

Remark 3. We have formulated Theorem 2 in terms of the uniform rule but the result is more general. The proof of Theorem 2 establishes that the following statement holds.

Assume that a rule on $\mathcal{P}^*$ can be characterized with a list of properties that include (ef) and (bm). Then, this rule can be extended to the general domain $\mathcal{P}$ (as with U) by adding $(cons)$ and (iic) to the list of properties characterizing the rule on $\mathcal{P}^*$.

5 Final Remarks

In this section we present some other properties that the uniform rule satisfies in the classical division problem. While some of them are satisfied by any extended uniform rule in our setting some others are not. Nevertheless, if we proceed by weakening such properties as we did with the principle of equal treatment of equals, any extended uniform rule satisfies the new formulations of the corresponding weaker principles.

A rule is *non-bossy* if whenever an agent receives the same allotment at two problems that are identical except for the preferences of this agent, then the allotments of all the other agents also coincide at the two problems. Formally,

NON-BOSSY For each problem P, each $i \in N$, and each $\succeq_i'$ such that $f_i(P) = f_i(P \backslash \succeq_i')$, then $f_j(P) = f_j(P \backslash \succeq_i')$ for all $j \in N \backslash \{i\}$.

A rule is own-peak monotonic if when the peak of an agent increases and the rest of the problem remains the same, this agent does not receive less.

OWN-PEAK MONOTONICITY For all $P, (P \backslash \succeq_i') \in \mathcal{P}$, $p_i' \leq p_i$ implies $f_i(P \backslash \succeq_i') \leq f_i(P)$.

A rule is tops-only when it depends only on the peaks of the preferences.

TOPS-ONLY For all $P, (P \backslash \succeq') \in \mathcal{P}$, $p_i = p_i'$ for all $i \in N$ implies $f(P) = f(P \backslash \succeq')$.

A rule satisfies maximality if the set of participants is always maximal according to set-wise inclusion.

MAXIMALITY For any $P \in \mathcal{P}$ and $T \subseteq N$ such that $c^f(P) \subsetneq T$, T is not an admissible coalition for P.

To show that any extended uniform rule on $\mathcal{P}$ induced by a monotonic and responsive order ρ on $\mathcal{N}$ satisfies the above properties is straightforward. We state this without proof as Proposition 1 below.

Proposition 1. *For each monotonic and responsive order ρ on $\mathcal{N}$, the extended uniform rule U^ρ is non-bossy, own-peak monotonic, tops-only, and satisfies maximality.*

We now introduce some properties that in the strong version (as in classical division problems) no extended uniform rule on $\mathcal{P}$ does satisfy. Nevertheless, a weaker version of them (obtained by weakening the properties as we did with equal treatment of equals) are satisfied by every extended uniform rule on $\mathcal{P}$. In all cases, when applied to classical division problems, the strong and the weak versions coincide.

The basic principle under envy-freeness is that no agent can strictly prefer the allotment received by another agent.

Strong envy freeness For each $P \in \mathcal{P}$ and each $i, j \in N$, $f_i(P) \succeq_i f_j(P)$.

We weaken this notion in two ways. First, we only require to compare allotments of participants (as in the case of ete). Second, we admit unfeasible envies (when agent i envies the allocation of agent j but agent i's allocation is not feasible for agent j).

Envy freeness For each $P \in \mathcal{P}$ and each pair of agents $i, j \in c^f$ such that $f_j(P) \succ_i f_i(P)$, then the vector of allotments $x = (x_k)_{k \in c^f}$, where $x_i = f_j(P)$, $x_j = f_i(P)$, and $x_k = f_k(P)$ for all $k \in c^f \backslash \{i, j\}$ has the property that $x \notin FA(P)$.

A rule is strongly individually rational from equal division if all agents receive an allotment that is at least as good as equal division.

Strongly individual rationality from equal division For each $P \in \mathcal{P}$ and each $i \in N$,

$$f_i(P) \succeq_i \frac{t}{n}.$$

We now weaken this principle by applying it only when the equal allotment is feasible.

Individual rationality from equal division For each $P \in \mathcal{P}$ for which $(\frac{t}{n}, ..., \frac{t}{n}) \in FA(P)$ then, for all $i \in N$,

$$f_i(P) \succeq_i \frac{t}{n}.$$

One-sided resource monotonicity says that if the good is scarce, an increase of the amount to be allotted should make all agents better off. Symmetrically, if the good is too abundant, a decrease of the amount to be allotted should make all agents better off.

Strong one-sided resource monotonicity For all $P, (P \backslash t') \in \mathcal{P}$ with the property that either $t \leq t' \leq \sum_{i \in N} p_i$ or $\sum_{i \in N} p_i \leq t' \leq t$ then, $f_i(P \backslash t') \succeq_i f_i(P)$ for all $i \in N$.

We weaken the principle by applying it only when, after changing the amount to be divided, the set of admissible coalitions does not change.

One-sided resource monotonicity For all $P, (P \backslash t') \in \mathcal{P}$ with the property that $A(P) = A(P \backslash t')$ and either $t \leq t' \leq \sum_{i \in N} p_i$ or $\sum_{i \in N} p_i \leq t' \leq t$ then, $f_i(P \backslash t') \succeq_i f_i(P)$ for all $i \in N$.

Proposition 2 *Let ρ be a monotonic and responsive order on $\mathcal{N}$. Then, the extended uniform rule U^ρ does not satisfy strong envy freeness, strong individual rationality from equal division, and strong one-sided resource monotonicity. Nevertheless, U^ρ satisfies envy freeness, individual rationality from equal division, and one-sided resource monotonicity.*

Proof See Subsection 6.3.

The proof of Proposition 2 establishes the following Corollary.

Corollary 1 *There is no rule on $\mathcal{P}$ that satisfies strong individual rationality from equal division. Moreover, let f be an efficient rule. Then, f neither satisfies strong envy freeness nor strong one-sided resource monotonicity on $\mathcal{P}$.*

In the classical division problem, efficient allocations are equivalent to same-sideness allocations; namely, $\sum_{i \in N} p_i \geq t$ implies that $x_i \leq p_i$ for all $i \in N$ and $\sum_{i \in N} p_i < t$ implies that $x_i \geq p_i$ for all $i \in N$. Nevertheless, this equivalence does not hold in the division problem under constraints. But first, we define same-sideness in our model.

Same-sideness Let $P \in \mathcal{P}$, $x = (x_i)_{i \in N} \in FA(P)$ and

$$c^x := \{i \in N \mid l_i \leq x_i \leq u_i\}.$$

The allocation x satisfies *same-sideness* if $\sum_{i \in c^x} p_i \geq t$ implies that $x_i \leq p_i$ for all $i \in c^x$ and $\sum_{i \in c^x} p_i < t$ implies that $x_i \geq p_i$ for all $i \in c^x$.

We can adapt the definition of maximality for an allocation x simply by replacing c^f by c^x in the definition of maximality for a rule f. Next result establishes the relationship between efficiency and same-sideness.

Proposition 3

(a) *If x satisfies maximality and same-sideness, then x is efficient.*

(b) *If x is efficient, then x satisfies same-sideness but it could fail maximality.*

The proof of Proposition 3 is straightforward and we omit it. Nevertheless let us clarify why efficiency does not imply maximality. Assume that S is admissible for x, $j \notin S$, $S \cup \{j\}$ is admissible for x, $\sum_{i \in S} p_i \leq t$, and $\sum_{i \in S \cup \{j\}} p_i > t$. Consider a profile of preferences where agents in S "prefer much more" to receive an allotment above their peaks than below, then x could be efficient even if it is not maximal. However, the reason of why an efficient allocation is not maximal is because the inclusion of an additional agent j transforms the problem from $\sum_{i \in S} p_i \leq t$ to $\sum_{i \in S \cup \{j\}} p_i > t$.[4]

[4]Namely, if S is admissible for x, $j \notin S$, $S \cup \{j\}$ is admissible for x, $\sum_{i \in S} p_i \leq t$, and $\sum_{i \in S \cup \{j\}} p_i \leq t$, then x is not efficient.

6 Proofs

We present the proofs of the main results of the paper.

6.1 Proof of Theorem 1

We first prove that U satisfies efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity on $\mathcal{P}^*$.

(1) U satisfies (ef). Fix a problem $P \in \mathcal{P}^*$. Assume that there exists $x = (x_i)_{i \in N} \in FA(P)$ with the property that $x_i \succeq_i U_i(P)$ for all $i \in N$. We prove that $x = U(P)$. Let $i \in N$ be arbitrary. We consider two cases.

1. $\sum_{j \in N} p_j < t$. Thus, $U_i(P) = \max\{p_i, \min\{u_i, \alpha\}\}$. In this case U coincides with the constrained uniform rule F studied in Bergantiños, Massó, and Neme (2012a). Using the same arguments used there to prove that F satisfies (ef) in the case $\sum_{j \in N} p_j < t$, we can prove that $x_i = U_i(P)$ holds.

2. $\sum_{j \in N} p_j \geq t$. Thus, $U_i(P) = \min\{p_i, \max\{l_i, \alpha\}\}$. We consider three cases.

   2.1. $U_i(P) = p_i$. Since $x_i \succeq_i U_i(P)$, it follows that $x_i = p_i$.

   2.2. $U_i(P) = \alpha < p_i$. Since $x_i \succeq_i U_i(P)$, by single-peakedness, $x_i \geq \alpha$. Suppose that $x_i > \alpha$. As $\sum_{j \in N} x_j = \sum_{j \in N} U_j(P) = t$, there exists $k \in N$ such that $x_k < U_k(P)$. We consider three cases.

   2.2.1. $U_k(P) = p_k$. Then, $x_k < p_k$, which contradicts that $x_k \succeq_k U_k(P)$.

   2.2.2. $U_k(P) = \alpha$ and $\alpha < p_k$. Then, $x_k < \alpha$, which contradicts, by single-peakedness, that $x_k \succeq_k U_k(P)$.

   2.2.3. $U_k(P) = l_k$. Since $x \in FA(P)$, $x_k = 0$ and hence $U_k(P) = l_k \succ_k 0 = x_k$, which contradicts $x_k \succeq_k U_k(P)$.

   Thus, $x_i = \alpha$ and hence, $x_i = U_i(P)$.

   2.3. $U_i(P) = l_i > \alpha$. Since $x_i \succeq_i U_i(P)$, $x_i \geq l_i$. Suppose that $x_i > l_i$. As $\sum_{j \in N} x_j = \sum_{j \in N} U_j(P) = t$, there exists $k \in N$ such that $x_k < U_k(P)$. Similarly to Case 2.2, we obtain a contradiction. Thus, $x_i = l_i$ and hence, $x_i = U_i(P)$.

(2) U satisfies (sp). Fix a problem $P \in \mathcal{P}^*$. Let $i \in N$ and $\succeq_i'$ on $[l_i, u_i]$ be arbitrary. We prove that $U_i(P) \succeq_i U_i(P \setminus \succeq_i')$. We consider two cases.

1. $\sum_{j \in N} p_j < t$. Thus, $U_i(P) = \max\{p_i, \min\{u_i, \alpha\}\}$. In this case U coincides with the constrained uniform rule F studied in Bergantiños, Massó, and Neme (2012a). Using the same arguments used there to prove that F satisfies (*sp*) in the case $\sum_{j \in N} p_j < t$, we can prove that $U_i(P) \succeq_i U_i(P \backslash \succeq'_i)$.

2. $\sum_{j \in N} p_j \geq t$. Thus, $U_i(P) = \min\{p_i, \max\{l_i, \alpha\}\}$. We consider three cases.

   The proofs of cases 2.1 $U_i(P) = p_i$ and 2.2 $U_i(P) = \alpha < p_i$ are similar to the proofs that F satisfies (*sp*) in Bergantiños, Massó, and Neme (2012a) when $\sum_{j \in N} p_j \geq t$, and $U_i(P) = p_i$ and $U_i(P) = \alpha < p_i$, respectively.

   We now consider the case 2.3 $U_i(P) = l_i > \alpha$. We consider three cases.

   2.3.1. $p'_i \geq p_i$. Then, $\sum_{j \in N \backslash \{i\}} p_j + p'_i \geq t$. Now $U_i(P \backslash \succeq'_i) = \min\{p'_i, \max\{l_i, \alpha'\}\}$. Since $\alpha < l_i \leq p_i \leq p'_i$, it follows that $\min\{p'_i, \max\{l_i, \alpha\}\} = l_i$. Hence, $\alpha' = \alpha$ and then, $U_i(P) = U_i(P \backslash \succeq'_i)$.

   2.3.2. $p'_i < p_i$ and $\sum_{j \in N \backslash \{i\}} p_j + p'_i \geq t$. The proof proceeds as in Case 2.3.1.

   2.3.3. $p'_i < p_i$ and $\sum_{j \in N \backslash \{i\}} p_j + p'_i < t$. Since $l_i \leq p'_i$,

   $$\sum_{j \in N \backslash \{i\}} p_j + l_i < t.$$

   Since $t = \sum_{j \in N \backslash \{i\}} U_j(P) + l_i$ and $p_j \geq U_j(P)$ for all $j \in N$,

   $$t \leq \sum_{j \in N \backslash \{i\}} p_j + l_i,$$

   a contradiction.

(3) By definition, U satisfies (*ete*).

(4) U satisfies (*bm*). We first prove (*bm.1*). Let P and $P' = P \backslash u'_k$ be as in the definition of (*bm.1*). We should prove that

$$U_i(P') \geq \min\{U_i(P), u'_i\} \text{ for each } i \in N.$$

We consider two cases.

1. $\sum_{j \in N} p_j < t$. Thus, $U_i(P) = \max\{p_i, \min\{u_i, \alpha\}\}$ for all $i \in N$. In this case U coincides with the constrained uniform rule F studied in Bergantiños, Massó, and Neme (2012a). Using the same arguments used there to prove that F satisfies (*sp*) in the case $\sum_{j \in N} p_j < t$, we can prove that (*bm.1*) holds in this case.

2. $\sum_{j\in N} p_j \geq t$. Thus, $U_i(P) = \min\{p_i, \max\{l_i, \alpha\}\}$ for all $i \in N$. We consider two cases.

   2.1. $\sum_{i\in N} p_i' < t$. Then, $p_k' = u_k' < p_k \leq u_k$. By definition of U, for each $i \in N\backslash\{k\}$, $U_i(P) \leq p_i = p_i' \leq U_i(P')$. Besides, $p_k' \leq U_k(P')$. Since $U_k(P') \in [l_k', u_k']$ and $u_k' = p_k'$, $U_k(P') = u_k'$ holds.

   2.2. $\sum_{i\in N} p_i' \geq t$. Then, $U_i(P') = \min\{p_i, \max\{l_i, \alpha'\}\}$ for all $i \in N\backslash\{k\}$. Since $p_k' \leq p_k$, $\alpha' \geq \alpha$. Therefore, for each $i \in N\backslash\{k\}$, $U_i(P') \geq U_i(P)$. To show that $U_k(P') \geq \min\{U_k(P), u_k'\}$ holds we consider two cases.

      2.2.1. $U_k(P) \leq u_k'$. Thus, $U_k(P) = U_k(P')$.

      2.2.2. $U_k(P) > u_k'$. Since $U_k(P) \leq p_k$ and preferences are single-peaked, $p_k' = u_k'$. If $\alpha \leq p_k$, then $U_k(P) = \max\{l_k, \alpha\}$. If $\alpha > p_k$, then $U_k(P) = p_k$. Since $\alpha' \geq \alpha$ we deduce in both cases ($\alpha \leq p_k$ and $\alpha > p_k$) that

$$U_k(P') = \min\{p_k', \max\{l_k, \alpha'\}\} = p_k' = u_k'.$$

We now prove (*bm.2*). Let P and $P' = P\backslash l_k'$ be as in the definition of (*bm.2*). We should prove that

$$U_i(P') \leq \max\{U_i(P), l_i'\} \text{ for each } i \in N.$$

We consider two cases.

1. $\sum_{j\in N} p_j < t$. Thus, $U_i(P) = \max\{p_i, \min\{u_i, \alpha\}\}$ for all $i \in N$. If $U_k(P) \geq l_k'$, then $U(P) = U(P')$, by the definition of U. Assume now that $U_k(P) < l_k'$. Since $p_k \leq U_k(P)$, $p_k < l_k'$. Hence $l_k' = p_k'$. We consider two cases.

   1.1. $\sum_{j\in N} p_j' > t$. Then, for all $i \in N\backslash\{k\}$,

$$U_i(P') = \min\{p_i', \max\{l_i', \alpha'\}\} \leq p_i' = p_i \leq U_i(P).$$

   Since $l_k' = p_k'$,

$$U_k(P') = \min\{p_k', \max\{l_k', \alpha'\}\} = l_k'.$$

   1.2. $\sum_{j\in N} p_j' \leq t$. Then, $U_i(P') = \max\{p_i', \min\{u_i, \alpha'\}\}$ for all $i \in N$. Since $p_k < p_k'$, $\alpha' \leq \alpha$. Thus, for all $i \in N\backslash\{k\}$,

$$\begin{aligned} U_i(P') &= \max\{p_i', \min\{u_i', \alpha'\}\} \\ &= \max\{p_i, \min\{u_i, \alpha'\}\} \\ &\leq \max\{p_i, \min\{u_i, \alpha\}\} \\ &= U_i(P). \end{aligned}$$

We consider two cases for agent k.

1.2.1. $\alpha \leq p_k$. Since $\alpha' \leq \alpha \leq p_k < l'_k = p'_k \leq u_k$,

$$U_k(P') = \max\{p'_k, \min\{u_k, \alpha'\}\} = p'_k = l'_k.$$

1.2.2. $\alpha > p_k$. Since $U_k(P) < l'_k \leq u'_k = u_k$,

$$U_k(P) = \max\{p_k, \min\{u_k, \alpha\}\} = \alpha.$$

Since $\alpha' \leq \alpha = U_k(P) < l'_k = p'_k \leq u_k$,

$$U_k(P') = \max\{p'_k, \min\{u_k, \alpha'\}\} = p'_k = l'_k.$$

2. $\sum_{j\in N} p_j \geq t$. Thus, $U_i(P) = \min\{p_i, \max\{l_i, \alpha\}\}$ for all $i \in N$. If $U_k(P) \geq l'_k$, then $U(P) = U(P')$, by the definition of U. Assume now that $U_k(P) < l'_k$. Notice that $p'_k \geq p_k$. Thus, $\sum_{j\in N} p'_j \geq t$. Hence, $U_i(P') = \min\{p'_i, \max\{l'_i, \alpha'\}\}$ for all $i \in N$. Then, $\alpha' \leq \alpha$. Proceeding as in Case 1.2, we deduce that $U_i(P') \leq U_i(P)$ for all $i \in N\backslash\{k\}$. We consider three cases for agent k.

2.1. $U_k(P) = l_k$. Then, $l_k \geq \alpha$. Since $\alpha' \leq \alpha \leq l_k = U_k(P) < l'_k \leq p'_k$,

$$U_k(P') = \min\{p'_k, \max\{l'_k, \alpha'\}\} = l'_k.$$

2.2. $U_k(P) = \alpha > l_k$. Then $p_k \geq \alpha$. Since $\alpha' \leq \alpha = U_k(P) < l'_k \leq p'_k$,

$$U_k(P') = \min\{p'_k, \max\{l'_k, \alpha'\}\} = l'_k.$$

2.3. $U_k(P) = p_k$. Then, $p_k \leq \alpha$. Since $p_k = U_k(P) < l'_k$, $p_k < p'_k = l'_k$. Since $\alpha' \leq \alpha$ we deduce that in the two possible cases (either $\alpha' \leq l'_k$ or $\alpha' > l'_k$) that

$$U_k(P') = \min\{p'_k, \max\{l'_k, \alpha'\}\} = l'_k.$$

We now prove that U is the unique rule satisfying the four properties on $\mathcal{P}^*$. We do it by proving the following five lemmata.

Lemma 1.1 *Let f be a rule satisfying (ef) and (bm) on $\mathcal{P}^*$ and let $P \in \mathcal{P}^*$. Then, $c^f(P) = N$.*

Proof Let $P \in \mathcal{P}^*$. Consider the problem $P^0 = (N, t, l^0, u^0, \succeq^0)$, where for all $i \in N$, $l^0_i = 0$ and $u^0_i = \max\{u_i, t\}$. Besides, for all $i \in N$, $\succeq^0_i$ coincides with $\succeq_i$ on $[l_i, u_i]$.

Obviously, $c^f(P) = N$ and by (ef), $\sum_{i\in N} f_i(P^0) = t$. Consider now the problem $P^0 \backslash l_1$. By (bm),

$$c^f\left(P^0 \backslash l_1\right) = c^f\left(P^0\right) = N.$$

Consider now the problem $P^0 \backslash \{l_1, u_1\}$ (if $u_1^0 = u_1$, then the next statement holds trivially). By (bm),

$$c^f\left(P^0 \backslash \{l_1, u_1\}\right) = c^f\left(P^0 \backslash l_1\right) = N.$$

Repeating this argument with agents $2, ..., n$, we obtain that $c^f(P) = N$. ■

An immediate consequence of Lemma 1.1 is that if f satisfies (ef), (bm), and (ete) on $\mathcal{P}^*$, then for all $P \in \mathcal{P}^*$ such that $\succeq_i = \succeq_j$ we have that $f_i(P) = f_j(P)$.

Next lemma is an extension of Lemma 1 in Ching (1994) for the classical division problem to the division problem under constraints.

Lemma 1.2 *Let f be a rule satisfying (ef) and (sp) on $\mathcal{P}^*$. Then, f is own-peak monotonic*[5]*.*

Proof Let $P, P \backslash \succeq_j' \in \mathcal{P}^*$ be such that $p_j' \leq p_j$. To obtain a contradiction, assume

$$f_j(P) < f_j(P \backslash \succeq_j'). \tag{3}$$

We consider two cases.

1. $\sum_{i\in N} p_i \leq t$. By (ef) and Proposition 3, $p_i \leq f_i(P)$ for all $i \in N$. Hence,

   $$p_j' \leq p_j \leq f_j(P) < f_j(P \backslash \succeq_j'),$$

   which implies, by single-peakedness, that $f_j(P) \succ_j' f_j(P \backslash \succeq_j')$, a contradiction with (sp).

2. $\sum_{i\in N} p_i > t$. By (ef) and Proposition 3,

   $$f_i(P) \leq p_i \text{ for all } i \in N. \tag{4}$$

   We consider two cases.

   2.1. $\sum_{i\neq j} p_i + p_j' \geq t$. By (ef) and Proposition 3, for all $i \neq j$, $f_i(P \backslash \succeq_j') \leq p_i$ and $f_j(P \backslash \succeq_j') \leq p_j'$. Hence, by (3),

   $$f_j(P) < f_j(P \backslash \succeq_j') \leq p_j' \leq p_j,$$

   which implies, by single-peakedness, that $f_j(P \backslash \succeq_j') \succ_j f_j(P)$, a contradiction with (sp).

[5] See Section 5 for a formal definition of own-peak monotonicity.

2.2. $\sum_{i\neq j} p_i + p'_j < t$. By (ef) and Proposition 3, for all $i \neq j$, $p_i \leq f_i(P\backslash \succeq'_j)$ and $p'_j \leq f_j(P\backslash \succeq'_j)$. Thus, $p'_j \leq f_j(P)$; otherwise, by (4),

$$t = \sum_{i\in N} f_i(P) < \sum_{i\neq j} p_i + p'_j$$

a contradiction. Hence,

$$p'_j \leq f_j(P) < f_j(P\backslash \succeq'_j),$$

which implies, by single-peakedness, that $f_j(P) \succ'_j f_j(P\backslash \succeq'_j)$, a contradiction with (sp). ■

Next lemma is an extension of Lemma 2 in Ching (1994) for the classical division problem to the division problem under constraints.

Lemma 1.3 *Let f be a rule satisfying (ef), (sp), and (bm) on $\mathcal{P}^*$. Then, for all $P \in \mathcal{P}^*$ and $j \in N$:*

(a) *If $p_j < f_j(P)$ and $\succeq'_j$ satisfies $0 \leq p'_j \leq f_j(P)$, then $f_j(P\backslash \succeq'_j) = f_j(P)$.*

(b) *If $f_j(P) < p_j$ and $\succeq'_j$ satisfies $f_j(P) \leq p'_j \leq t$, then $f_j(P\backslash \succeq'_j) = f_j(P)$.*

Proof Let f be an *(ef)* and (sp) rule, $P \in \mathcal{P}^*$ and $j \in N$.

(a) Assume $p_j < f_j(P)$ and let $\succeq'_j$ be such that $0 \leq p'_j \leq f_j(P)$. By (ef) and Lemma 1.1, Proposition 3 implies that

$$p_i \leq f_i(P) \text{ for all } i \in N. \tag{5}$$

Since $p'_j \leq f_j(P)$, (5) implies

$$\sum_{i\in c^f\backslash\{j\}} p_i + p'_j \leq \sum_{i\in c^f} f_i(P) = t.$$

We now show that $f_j(P\backslash \succeq'_j) = f_j(P)$. To obtain a contradiction, assume otherwise and consider two cases.

1. $f_j(P) < f_j(P\backslash \succeq'_j)$. Then,

$$p'_j \leq f_j(P) < f_j(P\backslash \succeq'_j),$$

which implies, by single-peakedness, that

$$f_j(P) \succ'_j f_j(P\backslash \succeq'_j),$$

contradicting (sp).

2. $f_j(P) > f_j(P\backslash \succeq'_j)$. We consider two cases.

   2.1. $f_j(P\backslash \succeq'_j) \geq p_j$. Then,
   $$p_j \leq f_j(P\backslash \succeq'_j) < f_j(P).$$
   By single-peakedness,
   $$f_j(P\backslash \succeq'_j) \succ_j f_j(P),$$
   contradicting (sp).

   2.2. $f_j(P\backslash \succeq'_j) < p_j$. Then, $p'_j > 0$ and
   $$f_j(P\backslash \succeq'_j) < p_j < f_j(P). \tag{6}$$
   By Lemma 1.1, $f_j(P\backslash \succeq'_j) \in [l_j, u_j]$. Let $\succeq''_j$ be such that $p''_j = p_j$ and
   $$f_j(P\backslash \succeq'_j) \succ''_j f_j(P). \tag{7}$$
   By Lemma 1.2, f is own-peak monotonic. Hence, $f_j(P\backslash \succeq''_j) = f_j(P)$. By (7),
   $$f_j(P\backslash \succeq'_j) \succ''_j f_i(P\backslash \succeq''_j),$$
   contradicting (sp).

(b) We omit the proof since it follows a symmetric argument to the one used to prove (a).
■

Lemma 1.4 *Let f be a rule satisfying (ef), (ete) and (bm) on $\mathcal{P}^*$. Assume $P\backslash \succeq'_{\{i,j\}} \in \mathcal{P}^*$ is such that $u_k = t$ for all $k \in N$ and* $\succeq'_i$ and $\succeq'_j$ *coincide on* $[\max\{l_i, l_j\}, t]$. *Then, it is not possible that simultaneously*
$$f_i(P\backslash \succeq'_{\{i,j\}}) < U_i(P\backslash \succeq'_{\{i,j\}})$$
and
$$f_j(P\backslash \succeq'_{\{i,j\}}) > U_j(P\backslash \succeq'_{\{i,j\}})$$
hold.

Proof We consider three cases.

1. $l_i = l_j$. By Lemma 1.1, $c^f(P\backslash \succeq'_{\{i,j\}}) = N$. Thus, i and j belong to $c^f(P\backslash \succeq'_{\{i,j\}})$. Since $\succeq'_i$ and $\succeq'_j$ coincide on $[\max\{l_i, l_j\}, t]$ and f and U satisfy (ete), the statement holds trivially.

2. $l_i < l_j$. Let $l_j^* = l_i$ and consider the preference $\succeq_j^*$ of agent j on $[l_j^*, u_j]$ that coincides with $\succeq_i'$ on $[l_i, u_i] = [l_j^*, u_j]$. Obviously,

$$P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\} \in \mathcal{P}^*.$$

By Lemma 1.1,

$$c^f(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) = N.$$

By *(ete)*,

$$\begin{aligned} f_i(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) &= f_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) \text{ and} \\ U_i(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) &= U_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}). \end{aligned} \tag{8}$$

Notice that the original problem $P \backslash \succeq_{\{i,j\}}'$ can be obtained from $P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}$ by increasing the lower bound of agent j from l_j^* to l_j. We consider three cases.

2.1. $l_j \leq \min \left\{ f_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}), U_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) \right\}$. Since the two rules satisfy (bm),

$$f(P \backslash \succeq_{\{i,j\}}') = f(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) \text{ and}$$

$$U(P \backslash \succeq_{\{i,j\}}') = U(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}).$$

Thus, the statement holds trivially.

2.2. $\min \left\{ f_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}), U_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) \right\} < l_j$
$\leq \max \left\{ f_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}), U_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) \right\}$.
Assume that[6]

$$f_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) < l_j \leq U_j(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}). \tag{9}$$

By (8)

$$f_i(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}) < U_i(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}). \tag{10}$$

Since the two rules satisfy (bm),

$$f_j(P \backslash \succeq_{\{i,j\}}') = l_j, \tag{11}$$

$$f_i(P \backslash \succeq_{\{i,j\}}') \leq f_i(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}), \text{ and} \tag{12}$$

$$U(P \backslash \succeq_{\{i,j\}}') = U(P \backslash \left\{ l_j^*, \succeq_i', \succeq_j^* \right\}). \tag{13}$$

[6]The proof of the other case is similar because we only use properties that the two rules satisfy.

Thus, by (11), (9), and (13),

$$f_j(P\backslash \succeq'_{\{i,j\}}) < U_j(P\backslash \succeq'_{\{i,j\}})$$

and, by (12), (10), and (13),

$$f_i(P\backslash \succeq'_{\{i,j\}}) < U_i(P\backslash \succeq'_{\{i,j\}}).$$

Hence, the statement holds.

2.3. $\max\left\{f_j(P\backslash \left\{l_j^*, \succeq'_i, \succeq_j^*\right\}), U_j(P\backslash \left\{l_j^*, \succeq'_i, \succeq_j^*\right\})\right\} < l_j$. Since the two rules satisfy (bm),

$$f_j(P\backslash \succeq'_{\{i,j\}}) = l_j \text{ and}$$
$$U_j(P\backslash \succeq'_{\{i,j\}}) = l_j,$$

which means that the statement holds trivially.

3. $l_i > l_j$. The proof proceeds as in Case 2, after changing the roles of i and j. ■

Lemma 1.5 *Let f be a rule satisfying (ef), (sp), (ete), and (bm) on $\mathcal{P}^*$. Then, $f = U$.*

Proof Let $P \in \mathcal{P}^*$ be arbitrary. We want to show that $f(P) = U(P)$. Since f is (bm) we can assume through the proof that $u_i \leq t$ for all $i \in N$. Otherwise, if $u_k > t$ for some agent k take $u'_k = t$; by (bm),

$$f\left(P\backslash u'_k\right) = f\left(P\right).$$

Assume first that $\sum_{i\in N} p_i \geq t$. By (ef) and Proposition 3, $f_i\left(P\right) \leq p_i$ for all $i \in N$.

Let $u_1^1 = t$ and consider any $\succeq_1^1$ defined on $[l_1, t]$ that coincides with $\succeq_1$ on $[l_1, u_1]$ and the peak of $\succeq_1$ and $\succeq_1^1$ (we call it p_1^1) coincide. For each $i \in N\backslash\{1\}$ define $\succeq_i^1 = \succeq_i$ and $u_i^1 = u_i$. Notice that P can be obtained from $P^1 = P\backslash\left\{u^1, \succeq^1\right\}$ by decreasing the upper bound of 1 from t to u_1. By (bm),

$$f_i\left(P\right) \geq \min\left\{f_i\left(P^1\right), u_i\right\} \text{ for each } i \in N. \tag{14}$$

Since $\sum_{i\in N} p_i^1 = \sum_{i\in N} p_i \geq t$ and f satisfies (ef), by Proposition 3, $f_i\left(P^1\right) \leq p_i$ for all $i \in N$. Since $p_i \leq u_i$ for all $i \in N$, by (14),

$$f_i\left(P\right) \geq f_i\left(P^1\right) \text{ for all } i \in N.$$

Hence,

$$f\left(P^1\right) = f\left(P\right).$$

Let $u_2^2 = t$ and consider any $\succeq_2^2$ defined on $[l_2, t]$ that coincides with $\succeq_2$ on $[l_2, u_2]$ and the peak of $\succeq_2$ and $\succeq_2^2$ coincide. For each $i \in N \backslash \{2\}$ define $\succeq_i^2 = \succeq_i$ and $u_i^2 = u_i^1$. Let $P^2 = (N, t, l, u^2, \succeq^2)$. Analogously to the previous case,

$$f\left(P^2\right) = f\left(P^1\right) = f(P).$$

Repeating this argument we obtain that $f(P^n) = f(P)$. Thus, we can assume that $u_i = t$ for all $i \in N$.

Without loss of generality assume that $p_1 \geq p_2 \geq ... \geq p_n$. To obtain a contradiction, assume that $U(P) \neq f(P)$. Then, there exists $i^1 \in N$ such that

$$U_{i^1}(P) < f_{i^1}(P) \leq p_{i^1} \leq p_1. \tag{15}$$

Step 1: Take $\succeq_{i^1}'$ defined on $[l_{i^1}, t]$ such that it coincides with $\succeq_1$ on $[\max\{l_1, l_{i^1}\}, t]$ and the peak of $\succeq_{i^1}'$ (denoted by p_{i^1}') is also p_1. Let $P^{i^1} = P \backslash \succeq_{i^1}'$. By (15), we can apply Lemma 1.3 (b) with $f = U$ and $j = i^1$. Then,

$$U_{i^1}(P^{i^1}) = U_{i^1}(P). \tag{16}$$

By Lemma 1.2, f is own-peak monotonic. Since $p_{i^1} \leq p_{i^1}'$,

$$f_{i^1}(P) \leq f_{i^1}(P^{i^1}). \tag{17}$$

By (16), (15), and (17)

$$U_{i^1}(P^{i^1}) < f_{i^1}(P^{i^1}).$$

Step 2: Then, there exists $i^2 \in N \backslash \{i^1\}$ such that

$$f_{i^2}(P^{i^1}) < U_{i^2}(P^{i^1}). \tag{18}$$

Take $\succeq_{i^2}'$ defined on $[l_{i^2}, t]$ such that $\succeq_{i^2}'$ coincides with $\succeq_{i^1}'$ on $[\max\{l_{i^1}, l_{i^2}\}, t]$ and the peak of $\succeq_{i^2}'$ (denoted by p_{i^2}') is also p_{i^1}' $(= p_1)$. Let $P^{i^1 i^2} = P^{i^1} \backslash \succeq_{i^2}'$. Since U satisfies (ef), by Proposition 3, $U_{i^2}(P^{i^1}) \leq p_{i^2} = p_{i^2}^{i^1}$. By (18), $f_{i^2}(P^{i^1}) < p_{i^2}$. Besides, $p_{i^1}' = p_1 \geq p_{i^2}$. By Lemma 1.3 (b) applied to f and $j = i^2$ we deduce that

$$f_{i^2}(P^{i^1 i^2}) = f_{i^2}(P^{i^1}). \tag{19}$$

By Lemma 1.2, f is own-peak monotonic. Since $p_{i^2} \leq p_{i^2}'$,

$$U_{i^2}(P^{i^1}) \leq U_{i^2}(P^{i^1 i^2}). \tag{20}$$

By (19), (18), and (20)

$$f_{i^2}(P^{i^1i^2}) < U_{i^2}(P^{i^1i^2}).$$

By Lemma 1.4,

$$f_{i^1}(P^{i^1i^2}) \leq U_{i^1}(P^{i^1i^2}).$$

Step 3: Then, there must exist $i^3 \in N\backslash\{i^1, i^2\}$ such that

$$U_{i^3}(P^{i^1i^2}) < f_{i^3}(P^{i^1i^2}). \tag{21}$$

Take $\succeq'_{i^3}$ defined on $[l_{i^3}, t]$ such that $\succeq'_{i^3}$ coincides with $\succeq'_{i^2}$ on $[\max\{l_{i^1}, l_{i^2}, l_{i^3}\}, t]$ and the peak of $\succeq'_{i^3}$ (denoted by p'_{i^3}) is also p'_{i^1} $(= p_1)$. Let $P^{i^1i^2i^3} = P^{i^1i^2}\backslash \succeq'_{i^3}$. By (21), $f_{i^3}(P^{i^1i^2}) < p_{i^3}$. Besides, $p'_{i^1} = p_1 \geq p_{i^3}$. By Lemma 1.3 (b) applied to $f = U$ and $j = i^3$ we deduce that

$$U_{i^3}(P^{i^1i^2i^3}) = U_{i^3}(P^{i^1i^2}). \tag{22}$$

By Lemma 1.2, f is own-peak monotonic. Since $p_{i^3} \leq p'_{i^3}$,

$$f_{i^3}(P^{i^1i^2}) \leq f_{i^3}(P^{i^1i^2i^3}). \tag{23}$$

By (22), (21), and (23)

$$U_{i^3}(P^{i^1i^2i^3}) < f_{i^3}(P^{i^1i^2i^3}).$$

By applying Lemma 1.4 to the pairs i^3, i^1 and i^3, i^2 we obtain

$$U_{i^1}(P^{i^1i^2i^3}) \leq f_{i^1}(P^{i^1i^2i^3}).$$

and

$$U_{i^2}(P^{i^1i^2i^3}) \leq f_{i^2}(P^{i^1i^2i^3}).$$

Continuing with this procedure, at Step n, we obtain that either

$$U_{i^n}\left(P\backslash \succeq'_N\right) < f_{i^n}\left(P\backslash \succeq'_N\right) \text{ and}$$

$$U_{i^j}\left(P\backslash \succeq'_N\right) \leq f_{i^j}\left(P\backslash \succeq'_N\right) \text{ for all } j \in N\backslash\left\{i^n\right\}$$

or else

$$f_{i^n}\left(P\backslash \succeq'_N\right) < U_{i^n}\left(P\backslash \succeq'_N\right) \text{ and}$$

$$f_{i^j}\left(P\backslash \succeq'_N\right) \leq U_{i^j}\left(P\backslash \succeq'_N\right) \text{ for all } j \in N\backslash\left\{i^n\right\}$$

In both cases we have a contradiction because

$$\sum_{i\in N} f_i\left(P\backslash \succeq'_N\right) = \sum_{i\in N} U_i\left(P\backslash \succeq'_N\right) = t.$$

Assume now that $\sum_{i\in N} p_i < t$. By (ef) and Proposition 3, $f_i(P) \geq p_i$ for all $i \in N$. We define $P^0 = P \backslash l^0$ where $l_i^0 = 0$ for all $i \in N$. Obviously, $P^0 \in \mathcal{P}^*$. Let $\mathcal{P}^0$ be the subdomain of $\mathcal{P}^*$ given by problems P with the property that $l_i = 0$ for all $i \in N$. In Theorem 1 of Bergantiños, Massó, and Neme (2012a) it is proved that there is a unique rule on $\mathcal{P}^0$ satisfying (ef), (sp), (ete), and upper bound monotonicity. This rule is called the constrained uniform rule and for all $i \in N$ it is defined as follows. For all $P \in \mathcal{P}^0$ and $i \in N$,

$$F_i(P) = \begin{cases} \min\{\beta, p_i\} & \text{if } \sum_{j\in N} p_j \geq t \\ \min\{\max\{\beta, p_i\}, u_i\} & \text{if } \sum_{j\in N} p_j < t, \end{cases}$$

where β is such that $\sum_{j\in N} F_j(P) = t$. Since ($bm.1$) coincides with upper bound monotonicity on $\mathcal{P}^0$, f coincides with F on $\mathcal{P}^0$. Thus, for all $i \in N$,

$$f_i\left(P^0\right) = \min\{\max\{\beta, p_i\}, u_i\},$$

where β satisfies $\sum_{j\in N} f_j\left(P^0\right) = t$. Besides, for all $i \in N$,

$$U_i\left(P^0\right) = \max\{p_i, \min\{u_i, \alpha\}\},$$

where α satisfies $\sum_{j\in N} f_j\left(P^0\right) = t$. It is immediate to see that for each δ,

$$\min\{\max\{\delta, p_i\}, u_i\} = \max\{p_i, \min\{u_i, \delta\}\}.$$

Thus, $\beta = \alpha$. Hence, for all $i \in N$, $f_i\left(P^0\right) = U_i\left(P^0\right)$.

Let $P^1 = P^0 \backslash l_1$. Since P^1 and P^0 belong to $\mathcal{P}^*$, by (bm),

$$\begin{aligned} f_1\left(P^1\right) &\leq \max\left\{f_1\left(P^0\right), l_1\right\} \text{ and} \\ f_i\left(P^1\right) &\leq \max\left\{f_i\left(P^0\right), 0\right\} \text{ for } i \neq 1. \end{aligned}$$

Since $f_1\left(P^0\right) = U_1\left(P^0\right) \geq p_1 \geq l_1$ we have that $f\left(P^1\right) = f\left(P^0\right)$.

Let $P^2 = P^1 \backslash l_2$. Since P^1 and P^2 belong to $\mathcal{P}^*$, by (bm),

$$\begin{aligned} f_2\left(P^2\right) &\leq \max\left\{f_2\left(P^1\right), l_2\right\} \text{ and} \\ f_i\left(P^2\right) &\leq \max\left\{f_i\left(P^1\right), 0\right\} \text{ for } i \neq 2. \end{aligned}$$

Since $f_2\left(P^1\right) = f_2\left(P^0\right) = U_2\left(P^0\right) \geq p_2 \geq l_2$ we have that $f\left(P^2\right) = f\left(P^1\right)$.

Repeating this argument for all $i = 3, ..., n$ we have that $f\left(P^n\right) = f\left(P^0\right)$. Since $P^n = P$, for all $i \in N$,

$$f_i(P) = f_i\left(P^0\right) = U_i\left(P^0\right) = \max\{p_i, \min\{u_i, \alpha\}\} = U_i(P).$$

This concludes the proof of Theorem 1's characterization.

We now prove that the four properties are independent.

- (ef) is independent of the other properties.

  We define the rule f^1 as follows. Let $P \in \mathcal{P}^*$. For each $i \in N$,

$$f_i^1(P) = median\{l_i, \alpha, u_i\},$$

  where α is such that $\sum_{i \in N} f_i(P) = t$. Then, f^1 satisfies (sp), (ete), and (bm) but fails (ef).

- (sp) is independent of the other properties.

  We define the rule f^2 as follows. Let $P \in \mathcal{P}^*$. For each $i \in N$,

$$f_i^2(P) = \begin{cases} p_i + \min\{\alpha, u_i - p_i\} & \text{if } \sum\limits_{i \in N} p_i < t \\ U_i(P) & \text{if } \sum\limits_{i \in N} p_i \geq t, \end{cases}$$

  where α is such that $\sum_{i \in N} f_i^2(P) = t$. Then, f^2 satisfies (ef), (ete), and (bm) but fails (sp).

- (ete) is independent of the other properties.

  We define f^3 as the priority rule given by the order $(1, 2, ..., n)$ applied to the set of efficient allocations. Namely, let $P \in \mathcal{P}^*$. We define f^3 formally, by considering separately the two following cases.

  1. $\sum_{i \in N} p_i \geq t$. Take k as the unique agent satisfying that $\sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} l_i \leq t < \sum_{i=1}^{k+1} p_i + \sum_{i=k+2}^{n} l_i$. For each $i \in N$,

$$f_i^3(P) = \begin{cases} p_i & \text{if } i \leq k \\ t - \sum_{i=1}^{k} p_i - \sum_{i=k+2}^{n} l_i & \text{if } i = k+1 \\ l_i & \text{if } i \geq k+2. \end{cases}$$

  2. $\sum_{i \in N} p_i < t$. Take k as the unique agent satisfying that $\sum_{i=1}^{k+1} p_i + \sum_{i=k+2}^{n} u_i \leq t < \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} u_i$. For each $i \in N$,

$$f_i^3(P) = \begin{cases} p_i & \text{if } i \leq k \\ t - \sum_{i=1}^{k} p_i - \sum_{i=k+2}^{n} u_i & \text{if } i = k+1 \\ u_i & \text{if } i \geq k+2. \end{cases}$$

Then, f^3 satisfies (*ef*), (*sp*), and (*bm*) but fails (*ete*).

- (*bm*) is independent of the other properties.

  We define the rule f^4 inspired by the Constant Equal Losses rule used in bankruptcy problems. Let $P \in \mathcal{P}^*$. For each $i \in N$,

$$f_i^4(P) = \begin{cases} \max\{u_i - \alpha, p_i\} & \text{if } \sum_{i\in N} p_i < t \\ \min\{\max\{l_i, u_i - \alpha\}, p_i\} & \text{if } \sum_{i\in N} p_i \geq t, \end{cases}$$

  where α is such that $\sum_{i\in N} f_i^4(P) = t$. Then, f^4 satisfies (*ef*), (*sp*), and (*ete*) but fails (*bm*).

6.2 Proof of Theorem 2

Let ρ be any monotonic and responsive order on $\mathcal{N}$. We prove that U^ρ satisfies (*ef*), (*sp*), (*ete*), (*bm*), (*cons)* and (*iic*) on $\mathcal{P}$.

(1) U^ρ satisfies (*cons*). Let $P \in \mathcal{P}$, $S \subseteq N$, and $i \in S$. We must prove that

$$U_i^\rho(P) = U_i^\rho\left(P^{U^\rho,S}\right)$$

where $P^{U^\rho,S} = \left(S, t - \sum\limits_{j\in c^{U^\rho}(P)\setminus S} U_j^\rho(P), (l_i)_{i\in S}, (u_i)_{i\in S}, (\succeq_i)_{i\in S}\right)$. We first prove that $c^{U^\rho}\left(P^{U^\rho,S}\right) = c^{U^\rho}(P) \cap S$. Suppose $c^{U^\rho}\left(P^{U^\rho,S}\right) \neq c^{U^\rho}(P) \cap S$. Since $c^{U^\rho}(P) \cap S \in A\left(P^{U^\rho,S}\right)$, we have $c^{U^\rho}\left(P^{U^\rho,S}\right) \rho \left(c^{U^\rho}(P) \cap S\right)$. Obviously, $c^{U^\rho}\left(P^{U^\rho,S}\right) \cup \left(c^{U^\rho}(P) \cap (N\setminus S)\right) \in A(P)$. By definition of U^ρ,

$$\left[c^{U^\rho}(P)\right] \quad \rho \quad \left[c^{U^\rho}\left(P^{U^\rho,S}\right) \cup \left(c^{U^\rho}(P) \cap (N\setminus S)\right)\right]. \tag{24}$$

Since ρ is responsive and $c^{U^\rho}\left(P^{U^\rho,S}\right) \rho \left[c^{U^\rho}(P) \cap S\right]$,

$$\left[c^{U^\rho}\left(P^{U^\rho,S}\right) \cup \left(c^{U^\rho}(P) \cap (N\setminus S)\right)\right] \quad \rho \quad \left[c^{U^\rho}(P) \cap S \cup \left(c^{U^\rho}(P) \cap (N\setminus S)\right)\right] = c^{U^\rho}(P)$$

which contradicts $c^{U^\rho}\left(P^{U^\rho,S}\right) \neq c^{U^\rho}(P) \cap S$ and (24). Thus, if $i \notin c^{U^\rho}(P) \cap S$, $U_i^\rho(P) = 0 = U_i^\rho\left(P^{U^\rho,S}\right)$ holds. Let $i \in c^{U^\rho}(P) \cap S$. We consider two cases.

1. $\sum\limits_{j\in c^{U^\rho}(P)} p_j \geq t$. Thus, $U_j^\rho(P) = \min\{p_j, \max\{l_l, \alpha\}\} \leq p_j$ for all $j \in c^{U^\rho}(P)$. Then,

$$\begin{aligned} \sum_{j\in c^{U^\rho}\left(P^{U^\rho,S}\right)} p_j &\geq \sum_{j\in c^{U^\rho}\left(P^{U^\rho,S}\right)} U_j^\rho(P) = \sum_{j\in c^{U^\rho}(P)\cap S} U_j^\rho(P) \\ &= t - \sum_{j\in c^{U^\rho}(P)\setminus S} U_j^\rho(P). \end{aligned}$$

Thus, $U_j^{\rho}\left(P^{U^{\rho},S}\right) = \min\{p_j, \max\{l_j, \alpha'\}\}$ for all $j \in c^{U^{\rho}}\left(P^{U^{\rho},S}\right)$. Since

$$\sum_{j \in c^{U^{\rho}}(P) \cap S} U_j^{\rho}\left(P^{U^{\rho},S}\right) = t - \sum_{j \in c^{U^{\rho}}(P) \setminus S} U_j^{\rho}(P) = \sum_{j \in c^{U^{\rho}}(P) \cap S} U_j^{\rho}(P),$$

it follows that $\alpha' = \alpha$. Hence, $U_i^{\rho}(P) = U_i^{\rho}\left(P^{U^{\rho},S}\right)$.

2. $\sum_{j \in c^{U^{\rho}}(P)} p_j < t$. The proof uses symmetric arguments to those already used in Case 1 and it is omitted.

(2) U^{ρ} satisfies (*iic*). Let $P, P' \in \mathcal{P}$ be such that $c^{U^{\rho}}(P) \in A(P') \subseteq A(P)$. By definition of $c^{U^{\rho}}(P)$, $c^{U^{\rho}}(P)\, \rho S$ for all $S \in A(P) \setminus c^{U^{\rho}}(P)$. Thus, $c^{U^{\rho}}(P)\, \rho S$ for all $S \in A(P') \setminus c^{U^{\rho}}(P)$. Hence, $c^{U^{\rho}}(P') = c^{U^{\rho}}(P)$.

(3) U^{ρ} satisfies (*ef*). Fix a problem $P \in \mathcal{P}$. Assume that there exists $x = (x_i)_{i \in N} \in FA(P)$ with the property that $x_i \succeq_i U_i^{\rho}(P)$ for all $i \in N$. We prove that $x = U^{\rho}(P)$. Since $x_i \succeq_i U_i^{\rho}(P)$ for all $i \in c^{U^{\rho}}(P)$, $x_i \in [l_i, u_i]$ for all $i \in c^{U^{\rho}}(P)$. Since ρ is monotonic and $c^{U^{\rho}}(P)$ is maximal in $FA(P)$, $x_i = 0 = U_i^{\rho}(P)$ for all $i \notin c^{U^{\rho}}(P)$. Since U^{ρ} satisfies (*cons*), for all $i \in c^{U^{\rho}}(P)$,

$$U_i^{\rho}(P) = U_i^{\rho}\left(P_{c^{U^{\rho}}(P)}\right).$$

Since $P_{c^{U^{\rho}}(P)} \in \mathcal{P}^*$, U^{ρ} coincides with U on $P_{c^{U^{\rho}}(P)}$ and U satisfies (*ef*) on $\mathcal{P}^*$, $x_i = U_i^{\rho}(P)$ for all $i \in c^{U^{\rho}}(P)$.

(4) U^{ρ} satisfies (*sp*). Let $P \in \mathcal{P}$, $i \in N$ and $\succeq_i'$ be as in the definition of (*sp*). We must prove that $U_i^{\rho}(P) \succeq_i U_i^{\rho}(P \backslash \succeq_i')$. Let us denote $P' = P \backslash \succeq_i'$. Obviously, $A(P') = A(P)$. Since U^{ρ} satisfies (*iic*), $c^{U^{\rho}}(P') = c^{U^{\rho}}(P)$. Then, $U_j^{\rho}(P) = U_j^{\rho}(P')$ for all $j \notin c^{U^{\rho}}(P)$. Thus, if $i \notin c^{U^{\rho}}(P), U_i^{\rho}(P) \succeq_i U_i^{\rho}(P')$. Since U^{ρ} satisfies (*cons*), for all $j \in c^{U^{\rho}}(P)$, $U_j^{\rho}(P) = U_j^{\rho}\left(P_{c^{U^{\rho}}(P)}\right)$ and $U_j^{\rho}(P') = U_j^{\rho}\left(P'_{c^{U^{\rho}}(P)}\right)$. Since $P_{c^{U^{\rho}}(P)} \in \mathcal{P}^*$, $P'_{c^{U^{\rho}}(P)} \in \mathcal{P}^*$, U^{ρ} coincides with U on $P_{c^{U^{\rho}}(P)}$ and $P'_{c^{U^{\rho}}(P)}$, and U satisfies (*sp*) on $\mathcal{P}^*$, we deduce that $U_i^{\rho}\left(P_{c^{U^{\rho}}(P)}\right) \succeq_i U_i^{\rho}\left(P'_{c^{U^{\rho}}(P)}\right)$.

(5) U^{ρ} satisfies (*ete*). It is obvious from the definition.

(6) U^{ρ} satisfies (*bm*). We first prove that U^{ρ} satisfies (*bm.1*). Let $P, (P \backslash u_k') \in \mathcal{P}$ be such that $u_k' < u_k$, and $c^{U^{\rho}}(P) \in A(P \backslash u_k')$. We denote $P' = P \backslash u_k'$. Since $u_k' < u_k$, $A(P') \subseteq A(P)$. Since U^{ρ} satisfies (*iic*), $c^{U^{\rho}}(P') = c^{U^{\rho}}(P)$. Then, $U_i^{\rho}(P) = 0 = U_i^{\rho}(P')$ for all $i \notin c^{U^{\rho}}(P)$. Thus, for all $i \notin c^{U^{\rho}}(P)$,

$$U_i^{\rho}(P') \geq \min\{U_i^{\rho}(P), u_i'\}.$$

Since U^ρ satisfies $(cons)$, for all $i \in c^{U^\rho}(P)$, $U_i^\rho(P) = U_i^\rho\left(P_{c^{U^\rho}(P)}\right)$ and $U_i^\rho(P') = U_i^\rho\left(P'_{c^{U^\rho}(P)}\right)$. Since $P_{c^{U^\rho}(P)} \in \mathcal{P}^*$, $P'_{c^{U^\rho}(P)} \in \mathcal{P}^*$, U^ρ coincides with U on $P_{c^{U^\rho}(P)}$ and $P'_{c^{U^\rho}(P)}$, and U satisfies $(bm.1)$ on $\mathcal{P}^*$, we deduce that for all $i \in c^{U^\rho}(P)$

$$\begin{aligned}
U_i^\rho(P') &= U_i^\rho\left(P'_{c^{U^\rho}(P)}\right) = U_i\left(P'_{c^{U^\rho}(P)}\right) \geq \min\left\{U_i\left(P_{c^{U^\rho}(P)}\right), u_i'\right\} \\
&= \min\left\{U_i^\rho\left(P_{c^{U^\rho}(P)}\right), u_i'\right\} \\
&= \min\left\{U_i^\rho(P), u_i'\right\}.
\end{aligned}$$

The proof that U^ρ satisfies $(bm.2)$ is similar to the proof that U^ρ satisfies $(bm.1)$ and it is omitted.

Let f be a rule satisfying (ef), (sp), (ete), (bm), $(cons)$ and (iic). We prove that there exists a monotonic and responsive order ρ on $\mathcal{N}$ for which $f = U^\rho$.

We first define a binary relation ρ on $\mathcal{N}$ as in Bergantiños, Massó, and Neme (2012b) Let $S, S' \in \mathcal{N}$. Three cases are possible.

1. $S \supset S'$. Then, set $S\rho S'$.

2. $S' \supset S$. Then, set $S'\rho S$.

3. There exist agents $j \in S \backslash S'$ and $j' \in S' \backslash S$. Consider any problem $P \in \mathcal{P}$ where $S, S' \subseteq N$ and for each $i \in N$, $l_i = p_i = u_i$, and

$$p_i = \begin{cases} \varepsilon & \text{if } i \in S \cap S' \\ \varepsilon^2 & \text{if } i \in S \backslash (S' \cup \{j\}) \\ t - \varepsilon|S \cap S'| - \varepsilon^2 |S \backslash (S' \cup \{j\})| & \text{if } i = j \\ \varepsilon^3 & \text{if } i \in S' \backslash (S \cup \{j'\}) \\ t - \varepsilon|S \cap S'| - \varepsilon^3 |S' \backslash (S \cup \{j'\})| & \text{if } i = j' \\ \varepsilon^4 & \text{if } i \in N \backslash (S \cup S'). \end{cases}$$

   We choose $\varepsilon > 0$ small enough to make sure that $0 < p_i < t$ for all $i \in N$ and $A(P) = \{S, S'\}$. By (ef), $c^f(P) \in \{S, S'\}$. Then, if $c^f(P) = S$ set $S\rho S'$ and if $c^f(P) = S'$ set $S'\rho S$.

Since f satisfies (iic), ρ is well defined because it does not depend on the particular chosen problem, namely given $P' \in \mathcal{P}$ such that $A(P') = \{S, S'\}$ we have that $c^f(P') = c^f(P)$. Thus, ρ is well defined.

By Lemma 11 and Lemma 13 in Bergantiños, Massó, and Neme (2012b), ρ is complete, antisymmetric, monotonic, responsive and transitive. By Lemma 12 in Bergantiños, Massó, and Neme (2012b) we have that $c^f(P)\,\rho S$ for all $S \in A(P) \backslash c^f(P)$. We should note that in the proofs of such lemmata the only properties of f used are (*ef*), (*cons*), and (*iic*).

Lemma 2.1 *Let f be a rule satisfying (ef), (sp), (ete), (bm), (cons), and (iic) and let ρ be its corresponding order defined as in Cases 1, 2, and 3 above. Then, $f = U^\rho$.*

Proof Let $P \in \mathcal{P}$ be arbitrary and suppose that f and ρ satisfy the hypothesis of Lemma 2.1. If $A(P) = \emptyset$, then $c^f(P) = c^{U^\rho}(P)$ and $f(P) = U^\rho(P) = (0, ..., 0)$. Assume $A(P) \neq \emptyset$. By (*ef*), $c^f(P)$ and $c^{U^\rho}(P)$ are non-empty. Since $S \in A(P_S)$ implies $S \in A(P)$, we have that $A(P_S) \subseteq A(P)$. In particular, $c^f(P) \in A(P_{c^f(P)}) \subseteq A(P)$. Hence, by (*iic*), $c^f(P_{c^f(P)}) = c^f(P)$. Since f satisfies (*cons*),

$$f_i(P) = \begin{cases} f_i(P_{c^f(P)}) & \text{if } i \in c^f(P) \\ 0 & \text{if } i \notin c^f(P). \end{cases} \tag{25}$$

Because $P_{c^f(P)} \in \mathcal{P}^*$ and f satisfies (*ef*), (*sp*), (*ete*), and (*bm*), by Theorem 1, for all $i \in c^f(P)$,

$$f_i(P_{c^f(P)}) = U_i^\rho(P_{c^f(P)}). \tag{26}$$

By (25) and (26), f coincides with U^ρ. ■

This concludes the proof of Theorem 2's characterization.

We now prove that the six properties are independent.

- (*ef*) is independent of the other properties.

  Let f^1 be defined as in the independence of the properties of Theorem 1. We extend f^1 to problems where N is not admissible as we did with the uniform rule. Namely, let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{1,\rho}$ as follows. For any $P \in \mathcal{P}$ and $i \in N$,

$$f_i^{1,\rho}(P) = \begin{cases} f_i^1(P_{c^{f^{1,\rho}}}) & \text{if } i \in c^{f^{1,\rho}} \\ 0 & \text{if } i \notin c^{f^{1,\rho}}, \end{cases}$$

  where $c^{f^{1,\rho}} \in A(P)$ and $c^{f^{1,\rho}} \rho S$ for all $S \in A(P) \backslash c^{f^{1,\rho}}$.

  It is not difficult to prove that $f^{1,\rho}$ satisfies all properties but (*ef*).

- (*sp*) is independent of the other properties.

  Let f^2 be defined as in the independence of the properties of Theorem 1. Let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{2,\rho}$ from f^2 as we did with $f^{1,\rho}$.

  It is not difficult to prove that $f^{2,\rho}$ satisfies all properties but (*sp*).

- (ete) is independent of the other properties.

  Let f^3 be defined as in the independence of the properties of Theorem 1. Let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{3,\rho}$ from f^3 as we did with $f^{1,\rho}$.

  It is not difficult to prove that $f^{3,\rho}$ satisfies all properties but (ete).

- (bm) is independent of the other properties.

  Let f^4 be defined as in the independence of the properties of Theorem 1. Let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{4,\rho}$ from f^4 as we did with $f^{1,\rho}$.

  It is not difficult to prove that $f^{4,\rho}$ satisfies all properties but (bm).

- (iic) is independent of the other properties.

  Let $N = \{1,2\}$ and ρ be such that

  $$\{1,2\}\,\rho\,\{1\}\,\rho\,\{2\}\,\rho\emptyset.$$

  We define f^5 as follows. For all $P \in \mathcal{P}$,

  $$f^5(P) = \begin{cases} (0,t) & \text{if } \{2\} \in A(P),\ N \notin A(P), \text{ and } t \geq 1 \\ U^\rho & \text{otherwise.} \end{cases}$$

  It is not difficult to prove that f^5 satisfies all properties but (iic).

- ($cons$) is independent of the other properties.

  Let ρ be a monotonic order on $\mathcal{N}$ but not responsive. We define $f^6 = U^\rho$.

  It is not difficult to prove that f^6 satisfies all properties but ($cons$).

6.3 Proof of Proposition 2

Let ρ be any monotonic and responsive order on $\mathcal{N}$.

(1) U^ρ satisfies envy freeness. Let $P \in \mathcal{P}$ and $i, j \in c^{U^\rho}$ be such that $U_i^\rho(P) \succ_j U_j^\rho(P)$. Since U^ρ satisfies consistency, $U_k^\rho(P) = U_k^\rho(P_{c^{U^\rho}})$ for all $k \in c^{U^\rho}$. Thus, we can assume that $c^{U^\rho} = N$; namely, $P \in \mathcal{P}^*$ and $U^\rho = U$. We consider two cases.

1. $\sum_{k\in N} p_k < t$. Then, $U_k(P) = \max\{p_k, \min\{u_k, \alpha\}\} \geq p_k$ for all $k \in N$. Since $U_i(P) \succ_j U_j(P)$, $U_j(P) = \min\{u_j, \alpha\} > p_j$. We consider two cases.

1.1. $\alpha < p_i$. Since $p_i \leq u_i$, $U_i(P) = p_i$. Since $U_i(P) \succ_j U_j(P) > p_j$, by single-peakedness, $U_i(P) < U_j(P)$. Then,

$$\alpha < p_i = U_i(P) < U_j(P) = \min\{u_j, \alpha\},$$

which is a contradiction.

1.2. $\alpha \geq p_i$. Since $p_i \leq u_i$, $U_i(P) = \min\{u_i, \alpha\}$. Since $U_i(P) \succ_j U_j(P)$, by single-peakedness,

$$\min\{u_i, \alpha\} = U_i(P) < U_j(P) = \min\{u_j, \alpha\}.$$

Since $U_i(P) = \alpha$ and $U_i(P) < U_j(P)$ are incompatible, we have that $U_i(P) = u_i < U_j(P)$. Thus, $U_j(P) \notin [l_i, u_i]$ which means that the allotment of j is not feasible for i.

2. $\sum_{k \in N} p_k \geq t$. Then, $U_k(P) = \min\{p_k, \max\{l_k, \alpha\}\} \leq p_k$ for all $k \in N$. Since $U_i(P) \succ_j U_j(P)$, $U_j(P) = \max\{l_j, \alpha\} < p_j$. We consider two cases.

2.1. $\alpha \leq p_i$. Since $l_i \leq p_i$, $U_i(P) = \max\{l_i, \alpha\}$. Since $U_i(P) \succ_j U_j(P)$, by single-peakedness,

$$\max\{l_i, \alpha\} = U_i(P) > U_j(P) = \max\{l_j, \alpha\}.$$

Since $U_i(P) = \alpha$ and $U_i(P) > U_j(P)$ are incompatible, we have that $U_i(P) = l_i > U_j(P)$. Thus, $U_j(P) \notin [l_i, u_i]$ which means that the allotment of j is not feasible for i.

2.2. $\alpha > p_i$. Since $l_i \leq p_i$, $U_i(P) = p_i$. Since $U_i(P) \succ_j U_j(P)$, by single-peakedness,

$$p_i = U_i(P) > U_j(P) = \max\{l_j, \alpha\},$$

a contradiction with $\alpha > p_i$.

(2) U^ρ satisfies individual rationality from equal division. Let $P \in \mathcal{P}$ be such that $(\frac{t}{n}, ..., \frac{t}{n}) \in FA(P)$. Thus, $l_i \leq \frac{t}{n} \leq u_i$ for all $i \in N$ and hence, $P \in \mathcal{P}^*$ and $U^\rho = U$. We consider two cases.

1. $\sum_{i \in N} p_i < t$. Then, $U_i(P) = \max\{p_i, \min\{u_i, \alpha\}\} \geq p_i$ for all $i \in N$. Assume that $\sum_{i \in N} u_i > t$ (otherwise $u_i = \frac{t}{n}$ for all $i \in N$ and the result holds trivially). Since for all $i \in N$, $\frac{t}{n} \leq u_i$ it follows that for all $i \in N$,

$$\max\left\{p_i, \min\{u_i, \tfrac{t}{n}\}\right\} = \max\left\{p_i, \tfrac{t}{n}\right\} \geq \tfrac{t}{n}.$$

Hence, $\alpha \leq \frac{t}{n}$. We consider two cases.

1.1. $p_i \geq \alpha$. Then, $U_i(P) = p_i \succeq_i \frac{t}{n}$.

1.2. $p_i < \alpha$. Then, $p_i < U_i(P) = \alpha \leq \frac{t}{n}$. By single-peakedness, $U_i(P) \succeq_i \frac{t}{n}$.

2. $\sum_{i \in N} p_i \geq t$. Then, $\min\{p_i, \max\{l_i, \alpha\}\} \leq p_i$ for all $i \in N$. Assume that $\sum_{i \in N} l_i < t$ (otherwise $l_i = \frac{t}{n}$ for all $i \in N$ and the result holds trivially). Since for all $i \in N$, $l_i \leq \frac{t}{n}$ it follows that for all $i \in N$,

$$\min\left\{p_i, \max\{l_i, \tfrac{t}{n}\}\right\} = \min\left\{p_i, \tfrac{t}{n}\right\} \leq \tfrac{t}{n} \text{ for all } i \in N.$$

Hence, $\alpha \geq \frac{t}{n}$. We consider two cases.

2.1. $p_i > \alpha$. Then, $\frac{t}{n} \leq \alpha = U_i(P) < p_i$. By single-peakedness, $U_i(P) \succeq_i \frac{t}{n}$.

2.2. $p_i \leq \alpha$. Then, $U_i(P) = p_i \succeq_i \frac{t}{n}$.

(3) U^ρ satisfies one-sided resource monotonicity. Let $P, (P \backslash t') \in \mathcal{P}$ be as in the definition of the property. Since $A(P) = A(P \backslash t')$ and U^ρ satisfies (iic), $c^{U^\rho}(P) = c^{U^\rho}(P \backslash t')$. Since U^ρ satisfies $(cons)$, and using similar arguments to those already used to prove that U^ρ is envy free, we can assume that $P \in \mathcal{P}^*$ and $U^\rho = U$. We consider two cases.

1. $\sum_{i \in N} p_i \leq t' \leq t$. Then, for all $i \in N$

$$\begin{aligned} U_i(P) &= \max\{p_i, \min\{u_i, \alpha\}\} \text{ and} \\ U_i(P \backslash t') &= \max\{p_i, \min\{u_i, \alpha'\}\}. \end{aligned}$$

Since $t' \leq t$, $\alpha' \leq \alpha$. Then, for all $i \in N$,

$$p_i \leq U_i(P \backslash t') \leq U_i(P).$$

By single-peakedness, $U_i(P \backslash t') \succeq_i U_i(P)$.

2. $t \leq t' \leq \sum_{i \in N} p_i$. The proof is symmetric to the prove of Case 1 and it is omitted.

(1$'$) U^ρ does not satisfy strong envy freeness. Let $P \in \mathcal{P}$ be such that $N = \{1, 2\}$, $t = 10$, $l = (7, 0)$, $u = (9, 9)$ and for each $x \in [1, 3]$ and $y \in [7, 9]$ we have that $y \succeq_2 x$. The set of feasible allocations is

$$FA(P) = \{(x_1, 10 - x_1) \mid x_1 \in [7, 9]\} \cup \{(0, 0)\}.$$

Since U^ρ is efficient, $U^\rho(P) \neq (0, 0)$, which means that U^ρ does not satisfy strong envy freeness.

$(2')$ U^ρ does not satisfy strong individual rationality from equal division. Let $P \in \mathcal{P}$ be such that $N = \{1,2\}$, $t = 10$, $l = (1,2)$, $u = (3,8)$ and $p_2 = 5$. The set of feasible allocations is

$$FA(P) = \{(x_1, 10 - x_1) \mid x_1 \in [2,3]\} \cup \{(0,0)\},$$

which means that U^ρ does not satisfy strong individual rationality from equal division.

$(3')$ U^ρ does not satisfy strong one-sided resource monotonicity. Let $P \in \mathcal{P}$ be such that $N = \{1,2,3\}$, $t = 10$, $t' = 14$, $l = (1,1,12)$, $u = (6,6,20)$, $p = (5,5,15)$, and for each $i \in \{1,2\}$ and each $x, y \in [1,6]$, $x \succeq_i y$ if and only if $|x-5| \leq |y-5|$. Now,

$$\begin{aligned} FA(P) &= \{(x_1, 10 - x_1, 0) \mid x_1 \in [4,6]\} \cup \{(0,0,0)\} \text{ and} \\ FA(P \backslash t') &= \{(x_1, 0, 14 - x_1) \mid x_1 \in [1,2]\} \cup \{(0, x_2, 14 - x_2) \mid x_2 \in [1,2]\} \\ &\quad \cup \{(1,1,12)\} \cup \{(0,0,14)\} \cup \{(0,0,0)\}. \end{aligned}$$

Since U^ρ is efficient, for each $i \in \{1,2\}$, $f_i(P) \in [4,6]$ and $f_i(P \backslash t') \leq 2$. Thus, U^ρ does not satisfy strong one-sided resource monotonicity. ■

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